

# Global stability of solutions to nonlinear wave equations

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## Abstract

We consider the problem of global stability of solutions to a class of semilinear wave equations with null condition in Minkowski space. We give sufficient conditions on the given solution which guarantees stability. Our stability result can be reduced to a small data global existence result for a class of semilinear wave equations with linear terms  $B^{\mu\nu}\partial_\mu\Phi(t,x)\partial_\nu\phi$ ,  $L^\mu(t,x)\partial_\mu\phi$  and quadratic terms  $h^{\mu\nu}(t,x)\partial_\mu\phi\partial_\nu\phi$  where the functions  $\Phi(t,x)$ ,  $L^\mu(t,x)$ ,  $h^{\mu\nu}(t,x)$  decay rather weakly and the constants  $B^{\mu\nu}$  satisfy the null condition. We show the small data global existence result by using the new approach developed by M. Dafermos and I. Rodnianski. In particular, we prove the global stability result under weaker assumptions than those imposed by S. Alinhac.

## 1 Introduction

In this paper, we study the behavior of solutions to the Cauchy problem

$$\begin{cases} \square w = \mathcal{N}(\partial w), \\ w(0, x) = \Phi_0(x) + \epsilon\phi_0(x), \quad \partial_t w(0, x) = \Phi_1(x) + \epsilon\phi_1(x) \end{cases} \quad (1)$$

in Minkowski space with initial data  $\Phi_i(x)$ ,  $\phi_i(x) \in C_0^\infty(\mathbb{R}^3)$ . The nonlinearity  $\mathcal{N}(\partial w)$  is assumed to satisfy the null condition, that is,  $\mathcal{N}(0) = D\mathcal{N}(0) = 0$  and the quadratic part of  $\mathcal{N}(\partial w)$  is  $A^{\alpha\beta}\partial_\alpha w \partial_\beta w$  with constant coefficients  $A^{\alpha\beta}$  such that  $A^{\alpha\beta}\xi_\alpha\xi_\beta = 0$  whenever  $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ .

In [1], S. Alinhac studied the stability of large solutions to the quasilinear wave equations

$$\begin{cases} \square w + g^{\alpha\beta\gamma}\partial_\gamma w \cdot \partial_{\alpha\beta} w = 0, \\ w(0, x) = \Phi_0(x), \quad \partial_t w(0, x) = \Phi_1(x) \end{cases}$$

in Minkowski space, where  $g^{\alpha\beta\gamma}$  are constants satisfying the null condition ( see [11]). More specifically, starting with a global solution  $\Phi(t, x) \in C^\infty(\mathbb{R}^{3+1})$ , consider the Cauchy problem with perturbed initial data  $(\Phi(0, x) + \epsilon\phi_0, \partial_t\Phi(0, x) + \epsilon\phi_1)$ . He showed that if  $\Phi$  satisfies the condition

$$|g^{ij\gamma}\partial_\gamma\Phi \cdot \xi_i\xi_j| \leq \alpha_0 \sum_{i=1}^3 |\xi_i|^2, \quad \sum_{|k|\leq 7} |\Gamma^k\partial\Phi| \leq C_0(1+t)^{-1}(1+|r-t|)^{-\frac{1}{2}} \quad (2)$$

for some positive constants  $\alpha_0 < 1$  and  $C_0$ , then the solution exists globally and is close to  $\Phi$ . Here  $\Gamma$  denotes the collection of Lorentz vector fields, see [10]. The problem of global stability of  $\Phi$  can be reduced to the following small data Cauchy problem

$$\begin{cases} \square\phi + g^{\alpha\beta\gamma}\partial_\gamma\phi \cdot \partial_{\alpha\beta}\phi + g^{\alpha\beta\gamma}\partial_\alpha\Phi\partial_{\beta\gamma}\phi + g^{\alpha\beta\gamma}\partial_{\beta\gamma}\Phi\partial_\alpha\phi = 0, \\ \phi(0, x) = \epsilon\phi_0(x), \quad \partial_t\phi(0, x) = \epsilon\phi_1(x) \end{cases}$$

with given function  $\Phi$  satisfying condition (2). The approach in [1] relies on the vector field method. In particular, S. Alinhac used the scaling vector field  $S = t\partial_t + r\partial_r$  with weights growing in  $t$  as commutators. The use of such weighted vector fields requires one to make the rather strong assumption that the given solution  $\Phi(t, x)$  decays uniformly in time  $t$  as in (2).

In this paper, we use the approach developed in [5], [22] to treat the problem of global stability of solutions to nonlinear wave equations. we use a new method for proving decay for linear problem, developed by M. Dafermos and I. Rodnianski in [5]. This new method avoids the use of vector fields containing positive weights in  $t$ , e.g.,  $S = t\partial_t + r\partial_r$ ,  $L_i = x_i\partial_t + t\partial_i$ . Traditionally, the vector fields from the set  $\Gamma$ , including  $S$ ,  $L_i$ , are used as multipliers or commutators. The new approach only commutes the equation with  $\partial_t$ ,  $x_i\partial_j - x_j\partial_i$  and allows us to obtain the stability results under conditions on  $\Phi$  weaker than those imposed by inequalities (2). We now describe the assumptions and the main results.

We denote  $(\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$  by  $\partial$  and  $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$  by  $\nabla$  under the coordinates  $(t, x_1, x_2, x_3)$ . We also use the null coordinates  $u = \frac{t-r}{2}$ ,  $v = \frac{t+r}{2}$  defined by the standard polar coordinates  $(t, r, \omega)$  in Minkowski space. The vector fields, used as commutators, are

$$Z = \{\Omega_{ij}, \partial_t = T\}, \quad \Omega_{ij} = x_i\partial_j - x_j\partial_i,$$

where Greek indices run from 0 to 3 while the Latin indices run from 1 to 3.

**Definition 1.** We call  $\Phi \in C^\infty(\mathbb{R}^{3+1})$  a  $(\delta, \alpha, t_0, R_1, C_0)$ -**weak wave** if

$$(i): |\partial\Phi(t, x)| \leq C_0, \quad t \leq t_0,$$

$$(ii): |\partial\Phi(t, x)| \leq C_0(1+r)^{-\frac{1}{2}}(1+(t-|x|)_+)^{-\frac{1}{2}-4\alpha}, \quad |x| \geq R_1, \quad t \geq t_0,$$

$$(iii): |\partial_v\Phi(t, x)| \leq C_0(1+r)^{-1-3\alpha}, \quad |x| \geq R_1, \quad t \geq t_0,$$

$$(iv): |\partial\Phi(t, x)| \leq \delta\alpha(1+r)^{-1-\alpha}, \quad |x| \leq R_1, \quad t \geq t_0$$

for some positive constants  $\delta, \alpha, t_0, R_1, C_0$ , where  $\partial_v = \partial_t + \partial_r$ . Here  $(t-|x|)_+ = \max\{0, t-|x|\}$ . Without loss of generality, we assume  $\alpha \leq \frac{1}{4}$  and  $R_1 \leq t_0$ .

**Remark 1.** Solution of a free wave equation in Minkowski space  $\square\Phi = 0$  with compactly supported initial data decays uniformly in time  $t$  and is always a  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave for some constants  $\delta, \alpha, t_0, R_1, C_0$ . We remark here that a weak wave does not have to decay uniformly in time  $t$  in the cylinder  $\{(t, x) | |x| \leq R_1\}$ .

In our argument, we estimate the decay of the solution with respect to the foliation  $\Sigma_\tau$ , defined as

$$S_\tau := \{u = u_\tau, v \geq v_\tau\}, \\ \Sigma_\tau := \{t = \tau, r \leq R\} \cup S_\tau,$$

where  $u_\tau = \frac{\tau-R}{2}$ ,  $v_\tau = \frac{\tau+R}{2}$ . The radius  $R$  is a to-be-fixed constant. The corresponding energy flux is

$$E[\phi](\tau) := \int_{r \leq R} |\partial\phi|^2 dx + \int_{S_\tau} (|\partial_v\phi|^2 + |\nabla\phi|^2) r^2 dv d\omega,$$

where  $\nabla$  denotes the induced covariant derivative on the sphere of radius  $r$ . We denote

$$E_0 = \sum_{|k| \leq 4} \int_{\mathbb{R}^3} |\partial Z^k \tilde{\phi}(0, x)|^2 dx,$$

where  $\tilde{\phi}(0, x) = \phi_0(x)$ ,  $\partial_t \tilde{\phi}(0, x) = \phi_1(x)$ . Here  $k$  stands for multiple indices, namely if  $k = (k_1, k_2)$ , then  $Z^k = \Omega^{k_1} T^{k_2}$ ,  $\Omega = \Omega_{ij}$ . And if  $k \leq \tilde{k}$ , then  $k_1 \leq \tilde{k}_1$ ,  $k_2 \leq \tilde{k}_2$ .

In addition to the assumption that the nonlinearity  $\mathcal{N}(\partial w)$  satisfies the null condition, we assume  $\mathcal{N}$  is smooth and

$$\mathcal{N}(\partial\Phi + \partial\phi) = \mathcal{N}(\partial\Phi) + A^{\mu\nu} \partial_\mu \Phi \partial_\nu \phi + \mathcal{N}^\mu(\partial\Phi) \partial_\mu \phi + \mathcal{N}^{\mu\nu}(\partial\Phi) \partial_\mu \phi \partial_\nu \phi + O(|\partial\phi|^3),$$

when  $\partial\phi$  is small. The coefficients  $\mathcal{N}^\mu(\partial\Phi)$ ,  $\mathcal{N}^{\mu\nu}(\partial\Phi)$  satisfy

$$\begin{aligned} |Z^\beta \mathcal{N}^\mu(\partial\Phi)| &\leq C(\partial\Phi) \sum_{|\beta'| \leq |\beta|} |Z^{\beta'} \partial\Phi|^{2+\alpha_0}, \quad \forall |\beta| \leq 4, \\ |Z^\beta \mathcal{N}^{\mu\nu}(\partial\Phi)| &\leq C(\partial\Phi) \sum_{|\beta'| \leq |\beta|} |Z^{\beta'} \partial\Phi|^{\alpha_0}, \quad \forall |\beta| \leq 4 \end{aligned} \quad (3)$$

for some positive constant  $\alpha_0$ . The constant  $C(\partial\Phi)$  depends only on  $\sum_{|\beta| \leq 4} \|Z^\beta \partial\Phi\|_{C^0}$ .

We now state our main results.

**Theorem 1.** *Suppose the nonlinearity  $\mathcal{N}(\partial w)$  satisfies the null condition and condition (3). Let  $\Phi \in C^\infty(\mathbb{R}^{3+1})$  be a solution of (1) when  $\epsilon = 0$ . Assume  $Z^k \Phi$  is  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave,  $\forall |k| \leq 4$ . Suppose the initial data  $\phi_0(x), \phi_1(x)$  are smooth and supported in  $\{|x| \leq R_0\}$ . Then there exists  $\delta_0 > 0$ , depending only on the constants  $A^{\alpha\beta}$ , and  $\epsilon_0 > 0$ , depending on  $E_0, R_0, A^{\alpha\beta}, \alpha_0, \alpha, t_0, R_1, C_0$ , such that for any  $\delta < \delta_0, \epsilon < \epsilon_0$ , there exists a unique global smooth solution  $w$  of equation (1) with the property that  $\exists$  positive constant  $R$ , depending on  $t_0, \alpha, \alpha_0, R_1, C_0, R_0$ , such that for the foliation  $\Sigma_\tau$  with radius  $R$ , the difference  $\phi = w - \Phi$  satisfies*

(1) *Energy decay*

$$E[\phi](\tau) \leq C E_0 \epsilon^2 (1 + \tau)^{-1 - \frac{1}{2}\alpha'}, \quad \alpha' = \min\left\{\frac{\alpha_0}{6}, \alpha\right\}.$$

(2) *Pointwise decay:*

$$\begin{aligned} |\phi| &\leq C \sqrt{E_0} \epsilon (1 + r)^{-1}, \\ \sum_{|\beta| \leq 2} |\partial^\beta \phi| &\leq C \sqrt{E_0} \epsilon (1 + r)^{-\frac{1}{2}} (1 + |t - r + R|)^{-\frac{1}{2} - \frac{\alpha'}{4}}, \quad \alpha' = \min\left\{\frac{\alpha_0}{6}, \alpha\right\}, \end{aligned}$$

where  $C$  depends on  $R, \alpha_0, \alpha, t_0, R_1, C_0$ .

**Remark 2.** *The weak decay of  $\partial\Phi$  in the spatial direction  $((1 + |x|)^{-\frac{1}{2}})$  excludes general cubic nonlinearities of  $\mathcal{N}(\partial\Phi)$  (cubic nonlinearities satisfying the null condition are allowed). However if condition (ii) in the definition of **weak wave**  $\Phi$  is improved to*

$$\sum_{|\beta| \leq 4} |\partial Z^\beta \Phi| \leq C_0 (1 + r)^{-\frac{1}{2} - \alpha} (1 + (t - |x|)_+)^{-\frac{1}{2} - 4\alpha},$$

then it is sufficient to assume

$$|Z^\beta \mathcal{N}^\mu(\partial\Phi)| \leq C(\partial\Phi) \sum_{|\beta'| \leq |\beta|} |Z^{\beta'} \partial\Phi|^2, \quad \forall |\beta| \leq 4.$$

This allows any cubic (or higher) nonlinearity of  $\mathcal{N}(\partial w)$ .

Since  $\Phi(t, x)$  solves (1) for  $\epsilon = 0$ , the problem of global stability of  $\Phi$  is then reduced to the following small data Cauchy problem

$$\begin{cases} \square\phi + N(\Phi, \phi) + L(\partial\phi) = F(\partial\phi), \\ \phi(0, x) = \epsilon\phi_0(x), \quad \phi_t(0, x) = \epsilon\phi_1(x), \end{cases} \quad (4)$$

where  $N(\Phi, \phi) = B^{\alpha\beta} \partial_\alpha \Phi \cdot \partial_\beta \phi$ ,  $L(\partial\phi) = L^\mu(t, x) \partial_\mu \phi$ . The nonlinearity  $F(\partial\phi)$  is of the form

$$\begin{aligned} F(\partial\phi) &= A^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + Q(\partial\phi) + \text{cubic and higher order terms of } \partial\phi, \\ Q(\partial\phi) &= h^{\mu\nu}(t, x) \partial_\mu \phi \partial_\nu \phi. \end{aligned}$$

Here  $A^{\mu\nu}$ ,  $B^{\mu\nu}$  are constants satisfying the null condition [11] and  $\Phi(t, x)$ ,  $L^\mu(t, x)$ ,  $h^{\mu\nu}(t, x)$  are given functions. For the stability problem, we have  $B^{\mu\nu} = -2A^{\mu\nu}$  and  $\phi = w - \Phi$ . However, it is of independent interest to consider the above small data Cauchy problem with linear terms  $N(\Phi, \phi)$ ,  $L^\mu \partial_\mu \phi$  and quadratic terms  $h^{\mu\nu}(t, x) \partial_\mu \phi \partial_\nu \phi$  where the functions  $\Phi(t, x)$ ,  $L^\mu(t, x)$ ,  $h^{\mu\nu}(t, x)$  decay rather weakly, given as follows:

For positive constants  $\delta$ ,  $\alpha$ ,  $t_0$ ,  $R_1$ ,  $C_0$ , we assume  $Z^k \Phi$  is  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave,  $\forall |k| \leq 4$  and

$$|\partial^2 Z^\beta \Phi| \leq C_0, \quad \forall |\beta| \leq 2.$$

Similarly, we assume

$$|\partial Z^\beta L^\mu| \leq C_0, \quad \forall |\beta| \leq 2.$$

For  $t \leq t_0$ , we assume  $Z^\beta L^\mu(t, x)$ ,  $Z^\beta h^{\mu\nu}(t, x)$  are bounded, that is,

$$|Z^\beta L^\mu(t, x)| + |Z^\beta h^{\mu\nu}(t, x)| \leq C_0, \quad \forall |\beta| \leq 4, \quad \forall t \leq t_0.$$

For  $t \geq t_0$ , we assume

$$|Z^\beta h^{\mu\nu}(t, x)| \leq C_0(1 + |x|)^{-\frac{3}{2}\alpha}, \quad \forall |\beta| \leq 4, \quad \forall t \geq t_0$$

and  $L^\mu(t, x)$  satisfies **one** of the following two conditions

$$|Z^\beta L^\mu(t, x)| \leq \delta \alpha (1 + |x|)^{-1-3\alpha}, \quad \forall |\beta| \leq 4, \quad \forall t \geq t_0 \quad (5)$$

or

$$|Z^\beta L^\mu(t, x)| \leq C_0(1 + |x|)^{-1-3\alpha}(1 + (t - |x|)_+)^{-\alpha}, \quad \forall |\beta| \leq 4, \quad \forall t \geq t_0. \quad (6)$$

Theorem 1 follows from:

**Theorem 2.** *Let  $\Phi(t, x)$ ,  $L^\mu(t, x)$ ,  $h^{\mu\nu}(t, x)$  be given smooth functions satisfying the above conditions.  $A^{\mu\nu}$ ,  $B^{\mu\nu}$  are constants satisfying the null condition. Assume the initial data  $\phi_0(x)$ ,  $\phi_1(x)$  are smooth and supported in  $\{|x| \leq R_0\}$ . Then there exists  $\delta_0 > 0$ , depending only on the constants  $B^{\mu\nu}$ , and  $\epsilon_0 > 0$ , depending on  $E_0, R_0$ ,  $A^{\mu\nu}$ ,  $B^{\mu\nu}$ ,  $\alpha, t_0, R_1, C_0$ , such that for any  $\delta < \delta_0$ ,  $\epsilon < \epsilon_0$ , there exists a unique global smooth solution  $\phi$  of the equation (4) with the property that  $\exists$  positive constant  $R$ , depending on  $t_0, \alpha, R_1, C_0, R_0$ , such that for the foliation  $\Sigma_\tau$  with radius  $R$ , the solution  $\phi$  satisfies*

(1) *Energy decay*

$$E[\phi](\tau) \leq C E_0 \epsilon^2 (1 + \tau)^{-1 - \frac{1}{2}\alpha}.$$

(2) *Pointwise decay*

$$\begin{aligned} |\phi| &\leq C \sqrt{E_0} \epsilon (1 + r)^{-1}, \\ \sum_{|\beta| \leq 2} |\partial^\beta \phi| &\leq C \sqrt{E_0} \epsilon (1 + r)^{-\frac{1}{2}} (1 + |t - r + R|)^{-\frac{1}{2} - \frac{\alpha}{4}}, \end{aligned}$$

where  $C$  depends on  $R$ ,  $\alpha_0$ ,  $\alpha$ ,  $t_0$ ,  $R_1$ ,  $C_0$ .

**Remark 3.** *Notice that  $\alpha$  can be arbitrarily small. The decay assumptions on  $L^\mu(t, x)$  (condition (5)) and  $h^{\mu\nu}(t, x)$  are sharp in the sense that there exists soliton solution (independent of time  $t$ ) to the linear wave equation if  $L^\mu(t, x)$  behaves like  $(1 + |x|)^{-1}$  and any nontrivial  $C^3$  solution of the equation*

$$\square \phi = \phi_t^2$$

*with compactly supported initial data blows up in finite time [9].*

**Remark 4.** We can also consider equation (4) with zeroth order linear term  $L_0(t, x)\phi$ , leading to the same conclusion provided that  $L_0(t, x)$  decays like  $(1 + |x|)^{-3-a}$ . Hence for the stability problem of large solution (Theorem 1), specific dependence on  $w$  of the nonlinearity  $\mathcal{N}(w)$  is also allowed.

**Remark 5.** For simplicity, we consider the equations in Minkowski space. However, as in [22], the same conclusion holds on curved background  $(\mathbb{R}^{3+1}, g)$  with metric  $g$  merely  $C^1$  close to the Minkowski metric and coinciding with the Minkowski metric outside the cylinder  $\{(t, x) ||x| \leq R\}$ .

**Remark 6.** It is not necessary to require that the initial data have compact support. The general assumption on the initial data can be that the following quantity

$$\sum_{|k| \leq 4} \int_{\mathbb{R}^3} r^{1+\alpha} |\partial Z^k \tilde{\phi}(0, x)|^2 dx, \quad \tilde{\phi}(0, x) = \phi_0(x), \quad \partial_t \tilde{\phi}(0, x) = \phi_1(x),$$

is finite.

The small data global existence result of nonlinear wave equations satisfying the null condition in Minkowski space was first obtained by D. Christodoulou [2] and S. Klainerman [11]. The approach of [11] used the vector field method, introduced by S. Klainerman in [10]. Various applications of the vector field method to nonlinear wave equations could be found in [12], [13], [15], [17], [19], [20]. In particular, the celebrated global nonlinear stability of Minkowski space has been proven by Christodoulou-Klainerman [3] and later by Lindblad-Rodnianski [14].

The main difficulty of considering nonlinear wave equation (4) with linear terms  $B^{\mu\nu} \partial_\mu \Phi(t, x) \partial_\nu \phi$ ,  $L^\mu(t, x) \partial_\mu \phi$  and quadratic terms  $h^{\mu\nu}(t, x) \partial_\mu \phi \partial_\nu \phi$  is the rather weak decay of the functions  $\Phi(t, x)$ ,  $L^\mu(t, x)$ ,  $h^{\mu\nu}(t, x)$ . Previous works have relied on the fact that these functions decay to zero uniformly in time  $t$ , which is not necessary in this context. In fact, we even allow these functions to stay static (independent of  $t$ ) in the cylinder  $\{(t, x) ||x| \leq R_1\}$ . Although we require  $\delta$  to be sufficiently small, which is the only smallness assumption here,  $\delta$  depends only on the constants  $B^{\mu\nu}$ .

Our argument here is similar to that in [22], which relies on a new approach, developed by M. Dafermos and I. Rodnianski in [5]. This new approach is a combination of an integrated local energy inequality and a p-weighted energy inequality in a neighborhood of the null infinity. However, due to the weak decay of  $\partial \Phi$ , we are not able to obtain the integrated local energy inequality and the p-weighted energy inequalities separately as in [22]. We thus consider these two inequalities together, see Proposition 1 in Section 2 for details.

The plan of this paper is as follows: we establish an integrated energy inequality in the whole space time and two p-weighted energy inequalities in Section 2. In Section 3, we use Proposition 1 to obtain the decay of the energy as well as the pointwise decay of the solution under appropriate bootstrap assumptions; in the last two sections, we close our bootstrap argument and conclude our main theorems.

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## 2 Notations and Preliminaries

In Minkowski space, we recall the energy-momentum tensor

$$\mathbb{T}_{\mu\nu}[\phi] = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\gamma \phi \partial_\gamma \phi.$$

Given a vector field  $X$ , we define the currents

$$J_\mu^X[\phi] = \mathbb{T}_{\mu\nu}[\phi] X^\nu, \quad K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi] \pi_{\mu\nu}^X,$$

where  $\pi_{\mu\nu}^X = \frac{1}{2}\mathcal{L}_X g_{\mu\nu}$  is the deformation tensor of the vector field  $X$ . Recall that

$$D^\mu J_\mu^X[\phi] = X(\phi)\square\phi + K^X[\phi].$$

Let  $n$  be the unit normal vector field to hypersurfaces,  $d\sigma$  the induced surface measure. We denote  $d\text{vol}$  as the volume form in Minkowski space. In null coordinates, we define the null infinity from  $\tau_1$  to  $\tau_2$  as follows

$$\mathcal{I}_{\tau_1}^{\tau_2} := \{(u, v, \omega) | u_{\tau_1} \leq u \leq u_{\tau_2}, v = \infty\}.$$

The corresponding energy flux is

$$I[\phi]_{\tau_1}^{\tau_2} := \int_{\mathcal{I}_{\tau_1}^{\tau_2}} ((\partial_u \phi)^2 + |\nabla \phi|^2) r^2 du d\omega \Big|_{v=\infty},$$

which is interpreted as a limit when  $v \rightarrow \infty$ . Denote

$$\tilde{E}[\phi](\tau) = E[\phi](\tau) + I[\phi]_0^\tau.$$

Here  $E[\phi](\tau)$  is defined in the introduction with the foliation  $\Sigma_\tau$ .

Taking a vector field

$$X = f(r)\partial_r,$$

for some function  $f(r)$ , consider the region bounded by the hypersurfaces  $\Sigma_{\tau_1}$  and  $\Sigma_{\tau_2}$ . By Stoke's formula, we have the identity

$$\begin{aligned} & \int_{\Sigma_{\tau_1}} J_\mu^X[\phi] n^\mu d\sigma - \int_{\Sigma_{\tau_2}} J_\mu^X[\phi] n^\mu d\sigma - \int_{\mathcal{I}_{\tau_1}^{\tau_2}} J_\mu^X[\phi] n^\mu d\sigma \\ &= \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} D^\mu J_\mu^X[\phi] d\text{vol} = \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \square\phi \cdot X(\phi) + K^X[\phi] d\text{vol}, \end{aligned} \quad (7)$$

where

$$K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi] \pi_{\mu\nu}^X = (\frac{1}{2}f' + r^{-1}f)(\partial_t \phi)^2 + (\frac{1}{2}f' - r^{-1}f)(\partial_r \phi)^2 - \frac{1}{2}f'|\nabla \phi|^2.$$

We use  $f'$  to denote  $\partial_r f$  throughout this paper.

Choose another function  $\chi$  of  $r$ , we have the equality

$$-\chi \partial^\gamma \phi \partial_\gamma \phi + \frac{1}{2} \square \chi \cdot \phi^2 = \frac{1}{2} D^\mu (\partial_\mu \chi \cdot \phi^2 - \chi \partial_\mu \phi^2) + \chi \phi \square \phi.$$

Add the above equality to both sides of (7). Define the current

$$\tilde{J}_\mu^X[\phi] = J_\mu^X[\phi] - \frac{1}{2} \partial_\mu \chi \cdot \phi^2 + \frac{1}{2} \chi \partial_\mu \phi^2.$$

Then we get

$$\begin{aligned} & \int_{\Sigma_{\tau_1}} \tilde{J}_\mu^X[\phi] n^\mu d\sigma - \int_{\Sigma_{\tau_2}} \tilde{J}_\mu^X[\phi] n^\mu d\sigma - \int_{\mathcal{I}_{\tau_1}^{\tau_2}} \tilde{J}_\mu^X[\phi] n^\mu d\sigma \\ &= \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (f \partial_r \phi + \phi \chi) \square \phi + (r^{-1}f + \frac{1}{2}f' - \chi)(\partial_t \phi)^2 \\ & \quad + (\chi - r^{-1}f + \frac{1}{2}f')(\partial_r \phi)^2 + (\chi - \frac{1}{2}f')|\nabla \phi|^2 - \frac{1}{2} \square \chi \cdot \phi^2. \end{aligned} \quad (8)$$

For  $X = T = \partial_t$  in (7), we have the energy inequality

$$\tilde{E}[\phi](\tau_2) \leq \tilde{E}[\phi](\tau_1) + 2 \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |\Box \phi| |\partial_t \phi| d\text{vol}. \quad (9)$$

We now fix the radius  $R$  of the foliation  $\Sigma_\tau$ . First under the assumptions in Theorem 2, we choose new positive constants  $\alpha'$ ,  $t'_0$ ,  $R'_1$  such that

$$|\partial_v Z^k \Phi| \leq \delta \alpha' (1 + |x|)^{-1-3\alpha'}, \quad t \geq t'_0, \quad |x| \geq R'_1, \quad |k| \leq 4, \quad (10)$$

$$|\partial Z^k \Phi| \leq 2\delta \alpha' (1 + |x|)^{-1-\alpha'}, \quad t \geq t'_0, \quad |x| \leq R'_1, \quad |k| \leq 4, \quad (11)$$

$$(1 + t'_0)^{\alpha'} \delta \alpha' \geq C_0. \quad (12)$$

In face, since  $Z^k \Phi$  is  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave  $\forall |k| \leq 4$ , choose  $\alpha' = \frac{1}{2}\alpha$  and  $R'_1$  large enough such that (10) holds. Then (11) and (12) are satisfied if  $t'_0$  is sufficiently large. The other conditions are also satisfied for these new constants  $\alpha'$ ,  $t'_0$ ,  $R'_1$ . Then the radius  $R$  can be fixed as

$$R = t'_0 + R_0,$$

where  $R_0$  is the radius of the support of the initial data. To avoid too many constants, we still use the constants  $\alpha$ ,  $R_1$ ,  $t_0$  to denote  $\alpha'$ ,  $R'_1$ ,  $t'_0$  respectively in the sequel.

The following several lemmas, which have been proven in [22], will be used later on.

**Lemma 1.** On  $S_\tau \cup \mathcal{I}_0^\tau$

$$r \int_{\omega} |\phi|^2 d\omega \leq \tilde{E}[\phi](\tau).$$

**Lemma 2.** If  $\phi$  is smooth, then

$$\int_{r \leq R} \left( \frac{\phi}{1+r} \right)^2 dx + \int_{S_\tau} \left( \frac{\phi}{1+r} \right)^2 r^2 dv d\omega \leq 6\tilde{E}[\phi](\tau).$$

**Corollary 1.** In the exterior region  $r \geq R$

$$\left| \int_{S_\tau} |\partial_v(r\phi)|^2 dv d\omega - \int_{S_\tau} \phi_v^2 r^2 dv d\omega \right| \leq 2\tilde{E}[\phi](\tau).$$

**Lemma 3.** Suppose  $f$  and  $\chi$  satisfy

$$|f| \leq C_1, \quad |\chi| \leq \frac{C_1}{1+r}, \quad |\chi'| \leq \frac{C_1}{(1+r)^2}$$

for some constant  $C_1$ , then

$$\left| \int_{\Sigma_\tau} \tilde{J}_\mu^X[\phi] n^\mu d\sigma \right| \leq 6C_1 \tilde{E}[\phi](\tau).$$

**Remark 7.** If  $\tilde{E}[\phi](\tau)$  is finite, all the above statements are also valid if we replace  $\tilde{E}[\phi](\tau)$  with  $E[\phi](\tau)$ .

Finally, we denote

$$\begin{aligned} \bar{\partial}\phi &:= (\partial\phi, \frac{\phi}{1+r}), \quad \overline{\partial}_v\phi := (\partial_v\phi, \nabla\phi), \\ g(p, \tau) &:= \int_{S_\tau} r^p |\partial_v \psi|^2 dv d\omega, \quad \bar{g}(p, \tau) := \int_{S_\tau} r^p |\overline{\partial}_v \psi|^2 dv d\omega, \\ G[\beta, p]_{\tau_1}^{\tau_2} &:= \int_{\tau_1}^{\tau_2} (1+\tau)^{-\beta} g(p, \tau) d\tau, \quad \bar{G}[\beta, p]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} (1+\tau)^{-\beta} \bar{g}(p, \tau) d\tau, \\ D^\beta[F]_{\tau_1}^{\tau_2} &:= \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |F|^2 (1+r)^{\beta+1} d\text{vol} \end{aligned}$$

for  $\forall p \geq 0$ ,  $\beta \in \mathbb{R}^1$ . Here  $\psi = r\phi$ ,  $\partial_v = \partial_t + \partial_r$  and  $\bar{\partial}\phi$  is a four dimensional vector with norm  $|\bar{\partial}\phi|^2 = |\partial\phi|^2 + \frac{\phi^2}{(1+r)^2}$ , similarly for  $\overline{\partial}_v\phi$ . Throughout this paper, we use the notation  $A \lesssim B$  for the inequality  $A \leq CB$  with some constant  $C$ , depending on  $R$ ,  $A^{\mu\nu}$ ,  $B^{\mu\nu}$ ,  $\alpha$ ,  $t_0$ ,  $R_1$ ,  $C_0$ .

### 3 Weighted Energy Inequalities

In this section, we use the multiplier method to establish an integrated local energy inequality and two p-weighted energy inequalities. The integrated local energy inequality was first proven by C. S. Morawetz in [18]. We follow the method in [4] to obtain the integrated local energy inequality here. In [5], M. Dafermos and I. Rodnianski introduced the p-weighted energy inequalities in a neighborhood of null infinity. These two estimates, which the decay of the energy flux  $E[\phi](\tau)$  relies on, were shown separately in [22]. Due to the weak decay of the functions  $\Phi(t, x)$ ,  $L^\mu(t, x)$ , we are not able to show these two estimates separately. We hence consider them together.

Consider the following linear wave equation

$$\begin{cases} \square\phi + N(\Phi, \phi) + L(\partial\phi) = F, \\ \phi(0, x) = \epsilon\phi_1(x), \quad \partial_t\phi(0, x) = \epsilon\phi_1(x), \end{cases} \quad (13)$$

where  $N(\Phi, \phi) = B^{\mu\nu}\partial_\mu\Phi \cdot \partial_\nu\phi$ ,  $L(\partial\phi) = L^\mu(t, x)\partial_\mu\phi$ . We have the following key estimates.

**Proposition 1.** *Suppose that  $\Phi$  is a  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave for positive constants  $\delta, \alpha, t_0, R_1, C_0$ . Assume the given functions  $L^\mu(t, x)$  satisfy*

$$|L^\mu(t, x)| \leq C_0, \quad t \leq t_0$$

and **one** of the following two conditions

$$|L^\mu(t, x)| \leq \delta\alpha(1 + |x|)^{-1-3\alpha}, \quad t \geq t_0,$$

or

$$|L^\mu(t, x)| \leq C_0(1 + |x|)^{-1-3\alpha}(1 + (t - |x|)_+)^{-\alpha}, \quad \forall t \geq t_0.$$

Suppose the constants  $\alpha, t_0, \delta, C_0$  obey the relation (12). Then there exists  $\delta_0 > 0$ , depending only on the constants  $B^{\mu\nu}$ , such that for all  $\delta < \delta_0$ , solution  $\phi$  of equation (13) has the following properties:

(1) *Integrated local energy estimate*

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{\alpha+1}} dx d\tau &\lesssim E[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + (1+\tau_1)^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \\ &\quad + (1+\tau_1)^{-1-2\alpha} \left( g(1+2\alpha, \tau_2) + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_\tau^{\tau_2} d\tau \right). \end{aligned} \quad (14)$$

(2) *Energy bound*

$$\begin{aligned} E[\phi](\tau_2) &\lesssim E[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + (1+\tau_1)^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \\ &\quad + (1+\tau_1)^{-1-2\alpha} \left( g(1+2\alpha, \tau_2) + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_\tau^{\tau_2} d\tau \right). \end{aligned} \quad (15)$$

(3) *p-weighted energy inequalities in a neighborhood of null infinity*

$$\begin{aligned} g(1, \tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau &\lesssim g(1, \tau_1) + \tau_1^{1-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \tau_1^{1-\alpha} E[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} (1+\tau)^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \\ &\quad + (1+\tau_1)^{-2\alpha} \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_\tau^{\tau_2} d\tau + (1+\tau_1)^{-2\alpha} g(1+2\alpha, \tau_1), \end{aligned} \quad (16)$$

$$\begin{aligned} g(1+2\alpha, \tau_2) + \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} &\lesssim g(1+2\alpha, \tau_1) + (1+\tau_1)^{1-\alpha} E[\phi](\tau_1) \\ &\quad + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_\tau^{\tau_2} d\tau + (1+\tau_1)^{1+\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2}. \end{aligned} \quad (17)$$



**Remark 8.** We mention here that variants and generalizations of estimate (14) can also be found in [16], [17].

The following corollary will be used to derive the energy decay estimates when commuting the equation with the vector fields  $Z$ .

**Corollary 2.** Assume the given functions  $\Phi(t, x)$ ,  $L^\mu(t, x)$  and the constant  $\delta$  satisfy the conditions in the above proposition. Then for solution  $\phi$  of (13), we have estimates for  $N = B^{\mu\nu} \partial_\mu \Phi \cdot \partial_\nu \phi$ ,  $L = L^\mu \partial_\mu \phi$

$$D^{2\alpha}[N]_{\tau_1}^{\tau_2} + D^{2\alpha}[L]_{\tau_1}^{\tau_2} \lesssim E[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + (1 + \tau_1)^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \\ + (1 + \tau_1)^{-1-2\alpha} \left( g(1 + 2\alpha, \tau_2) + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \right).$$

Under appropriate bootstrap assumptions on the nonlinearity  $F$ , the above inequalities lead to decay of the energy flux  $E[\phi](\tau)$ . We discuss the integrated local energy inequality and the p-weighted energy inequalities separately. And then combine them together to prove the above proposition. The following two lemmas will be used frequently. First define

$$A = 10 \sup_{\mu, \nu} \{|B^{\mu\nu}|\}.$$

**Lemma 4.** Let  $N = B^{\mu\nu} \partial_\mu \Phi \partial_\nu \phi$ . Then

$$|rN| \leq A (|\partial\Phi| |\overline{\partial}_v \psi| + |\partial_v \Phi| |\partial\psi| + |\partial\Phi| |\phi|), \quad \psi = r\phi.$$

*Proof.* By our notations

$$rN = rB^{\mu\nu} \partial_\mu \Phi \cdot \partial_\nu \phi = B^{\mu\nu} \partial_\mu \Phi \cdot \partial_\nu \psi - B^{\mu\nu} \partial_\mu \Phi \partial_\nu r \cdot \phi.$$

The lemma then follows from the fact that  $B^{\mu\nu}$  satisfies the null condition and the inequality  $|\partial r| \leq 1$ .  $\square$

**Lemma 5** (Gronwall's Inequality). Suppose  $A(\tau)$ ,  $E(\tau)$  are nonnegative functions on  $[\tau_1, \tau_2]$ . Assume that  $E(\tau)$  is nondecreasing on this interval and  $\beta$  is a positive number. If

$$A(\tau) \leq E[\tau] + C \int_{\tau_1}^{\tau} (1 + s)^{-1-\beta} A(s) ds, \quad \forall \tau \in [\tau_1, \tau_2],$$

then

$$A(\tau) \leq \exp(C\beta^{-1}(1 + \tau_1)^{-\beta}) E(\tau), \quad \forall \tau \in [\tau_1, \tau_2].$$

*Proof.* See [21].  $\square$

### 3.1 Integrated Local Energy Inequality

We follow the idea used in [4] by choosing appropriate functions  $f$  and  $\chi$  such that the coefficients on the right hand side of (8) are positive. The left hand side can be controlled by the energy flux  $\tilde{E}[\phi]$  by Lemma 3. We thus end up with an integrated energy inequality in the whole space time. We now discuss this in detail.

Take

$$f = \beta - \frac{\beta}{(1+r)^\alpha}, \quad \chi = r^{-1}f, \quad \beta = \frac{2}{\alpha}.$$

Notice that

$$\frac{(1+r)^\alpha - 1}{r} \geq \frac{\alpha}{1+r}.$$

We have

$$\begin{aligned} r^{-1}f + \frac{1}{2}f' - \chi &= \chi - r^{-1}f + \frac{1}{2}f' = \frac{1}{(1+r)^{1+\alpha}}, \\ \chi - \frac{1}{2}f' &= \frac{\beta((1+r)^\alpha - 1)}{r(1+r)^\alpha} - \frac{1}{(1+r)^{1+\alpha}} \geq \frac{1}{(1+r)^{\alpha+1}}, \\ -\frac{1}{2}\square\chi &= \frac{\alpha+1}{r(1+r)^{2+\alpha}}. \end{aligned}$$

Hen the energy inequalities (8), (9) together with Lemma 3 imply that

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} \leq 12\beta\tilde{E}[\phi](\tau_1) + 13\beta \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |F - N - L||\bar{\partial}\phi| d\text{vol}. \quad (18)$$

To proceed, we have to estimate the linear terms  $N(\Phi, \phi)$ ,  $L(\partial\phi)$ . We first consider the case  $\tau_2 \geq \tau_1 \geq t_0$ . For  $\tau \geq t_0$ , notice that on  $\Sigma_\tau$

$$C_0(1+|x|)^{-\alpha}(1+(t-|x|)_+)^{-\alpha} \leq C_0(1+\tau)^{-\alpha} \leq C_0(1+t_0)^{-\alpha} \leq \delta\alpha$$

by the inequality (12)(we have assumed this inequality in Proposition 1). Hence under the conditions on the functions  $L^\mu(t, x)$  in Proposition 1, we always have

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |L(\partial\phi)||\bar{\partial}\phi| d\text{vol} \leq \delta\alpha \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (1+r)^{-1-\alpha} |\bar{\partial}\phi|^2 d\text{vol}.$$

For  $N(\Phi, \phi)$ , we consider it inside and outside the cylinder  $\{|x| \leq R_1\}$  separately. When  $r = |x| \leq R_1$ , the null structure of  $N(\Phi, \phi)$  is not necessary. We has to rely on the smallness of  $\delta$ . Since  $\Phi$  is a weak wave, condition (iv) of Definition 1 implies that

$$|N| = |B^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\phi| \leq A\delta\alpha(1+r)^{-1-\alpha}|\partial\phi|.$$

For  $r \geq R_1$ , the null structure of  $N(\Phi, \phi)$  is of particular importance. By Lemma 4, it suffices to estimate the three terms  $r^{-1}|\partial\Phi||\bar{\partial}_v\psi||\bar{\partial}\phi|$ ,  $r^{-1}|\partial_v\Phi||\partial\psi||\bar{\partial}\phi|$ ,  $r^{-1}|\partial\Phi||\phi||\bar{\partial}\phi|$ . Without loss of generality, assume  $R_1 \geq 1$ . For the second term, inequality (10) shows that

$$|\partial_v\Phi||r^{-1}\partial\psi||\bar{\partial}\phi| \leq 2\delta\alpha(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2.$$

On  $\Sigma_\tau \cap \{|x| \geq R_1 \geq 1\}$ , for the first term, we have

$$\begin{aligned} r^{-1}|\partial\Phi||\bar{\partial}_v\psi||\bar{\partial}\phi| &\leq C(1+r)^{-\frac{3}{2}}(1+\tau)^{-\frac{1}{2}-4\alpha}|\bar{\partial}_v\psi||\bar{\partial}\phi| \\ &\leq \delta\alpha(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2 + C(1+\tau)^{-1-8\alpha}r^{-2+\alpha}|\bar{\partial}_v\psi|^2. \end{aligned}$$

Here we denote  $C$  as a constant depending on  $\alpha$ ,  $R = t_0 + R_0$ ,  $A^{\alpha\beta}$ ,  $B^{\alpha\beta}$ ,  $C_0$ ,  $\delta$ . Similarly for the third term, we have

$$r^{-1}|\partial\Phi||\phi||\bar{\partial}\phi| \leq \delta\alpha(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2 + C(1+\tau)^{-1-8\alpha}r^{-2+\alpha}|\phi|^2.$$

It remains to control  $r^{-2+\alpha}|\phi|^2$ . We use the Hardy's inequality outside the cylinder  $\{|x| \leq R\}$ . By Lemma 1, we have

$$\begin{aligned} \int_{\omega} |\psi|^2(\tau, v, \omega) d\omega &\leq C \int_{\omega} |\psi|^2(\tau, v_\tau, \omega) d\omega + C \left( \int_{v_\tau}^v \int_{\omega} |\partial_v\psi| d\omega dv \right)^2 \\ &\leq C\tilde{E}[\phi](\tau) + C \int_{v_\tau}^v \int_{\omega} r^{1+2\alpha} |\partial_v\psi|^2 d\omega dv \int_{v_\tau}^v r^{-1-2\alpha} dv \\ &\leq C\tilde{E}[\phi](\tau) + Cg(1+2\alpha, \tau), \quad (\tau, v, \omega) \in S_\tau, \end{aligned} \quad (19)$$

where  $v_\tau = \frac{R+\tau}{2}$ ,  $v = \frac{r+t}{2}$ . Hence for all  $p \leq 1 + 2\alpha$

$$\int_{S_\tau} r^{p-3\alpha} \phi^2 dv d\omega = \int_{v_\tau}^\infty r^{p-2-3\alpha} \int_\omega |\psi|^2 d\omega dv \leq C\tilde{E}[\phi](\tau) + Cg(1+2\alpha, \tau). \quad (20)$$

On the other hand, Lemma 2 shows that

$$\int_{S_\tau} \phi^2 dv d\omega \leq C\tilde{E}[\phi](\tau).$$

Interpolate with (20) for  $p = 1 + 2\alpha$ . We derive

$$\begin{aligned} \int_{S_\tau} r^\alpha \phi^2 dv d\omega &\leq C\tilde{E}[\phi](\tau)^{1-\gamma} \left( \tilde{E}[\phi](\tau) + g(1+2\alpha, \tau) \right)^\gamma \\ &\leq C\tilde{E}[\phi](\tau) + C\tilde{E}[\phi](\tau)^{1-\gamma} g(1+2\alpha, \tau)^\gamma, \end{aligned}$$

where  $\gamma = \frac{\alpha}{1-\alpha}$ . This gives estimates for  $\phi^2$  outside the cylinder  $\{|x| \leq R\}$ .

In the region  $R_1 \leq r \leq R$ , using Sobolev embedding and Lemma 1, we get

$$\int_\omega \phi^2 d\omega \leq C \int_\omega \phi^2 d\omega \Big|_{r=R} + C \int_{r \leq R} |\partial_r \phi|^2 dx \leq C\tilde{E}[\phi](\tau).$$

Therefore we can estimate  $r^{-2+\alpha}|\phi|^2$  outside the cylinder  $\{|x| \leq R_1\}$  as follows

$$\begin{aligned} \int_{\{r \geq R_1\} \cap \Sigma_\tau} r^{-2+\alpha} |\phi|^2 d\sigma &= \int_{R_1 \leq r \leq R} r^{-2+\alpha} \phi^2 dx + \int_{S_\tau} r^\alpha \phi^2 dv d\omega \\ &\leq C\tilde{E}[\phi](\tau) + C\tilde{E}[\phi](\tau)^{1-\gamma} g(1+2\alpha, \tau)^\gamma. \end{aligned}$$

Inside the cylinder  $\{|x| \leq R_1\}$ , we use the assumption that  $\partial\Phi$  is small. Hence combining the above estimates, we can bound the linear term  $(|N(\Phi, \phi)| + |L(\partial\phi)|)|\bar{\partial}\phi|$  in (18) as follows

$$\begin{aligned} \beta \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (|N| + |L|) |\bar{\partial}\phi| d\text{vol} &\leq A\delta \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} + C \int_{\tau_1}^{\tau_2} \frac{\tilde{E}[\phi](\tau)}{(1+\tau)^{1+8\alpha}} d\tau + C\bar{G}[1+8\alpha, \alpha]_{\tau_1}^{\tau_2} \\ &\quad + C \left( \int_{\tau_1}^{\tau_2} (1+\tau)^{-1-2\alpha} \tilde{E}[\phi](\tau) d\tau \right)^{1-\gamma} \left( \int_{\tau_1}^{\tau_2} (1+\tau)^{-7+4\alpha} g(1+2\alpha, \tau) d\tau \right)^\gamma \\ &\leq A\delta \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} + C \int_{\tau_1}^{\tau_2} \frac{\tilde{E}[\phi](\tau)}{(1+\tau)^{1+2\alpha}} d\tau \\ &\quad + CG[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + C\bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2}, \end{aligned}$$

where we used Hölder's inequality and Jensen's inequality

$$a^{1-\gamma} b^\gamma \leq (1-\gamma)a + \gamma b, \quad \forall a, b > 0.$$

For the inhomogeneous term  $|F||\bar{\partial}\phi|$  in (18), we have

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |F| |\bar{\partial}\phi| d\text{vol} \leq \delta\alpha \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} + CD^\alpha [F]_{\tau_1}^{\tau_2}$$

If we choose

$$\delta_0 = \frac{A}{100},$$

then for all  $\delta < \delta_0$ , inequality (18) implies that

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} &\lesssim \tilde{E}[\phi](\tau_1) + D^\alpha [F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \frac{\tilde{E}[\phi](\tau)}{(1+\tau)^{1+2\alpha}} d\tau \\ &\quad + G[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2}. \end{aligned}$$

Similarly, the energy inequality (9) shows that

$$\tilde{E}[\phi](\tau_2) \lesssim \tilde{E}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \frac{\tilde{E}[\phi](\tau)}{(1+\tau)^{1+2\alpha}} d\tau + G[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2} + D^\alpha[F]_{\tau_1}^{\tau_2}.$$

We Gronwall's inequality to control the second term on the right hand side of the above inequality. We thus have

$$\tilde{E}[\phi](\tau_2) \lesssim \tilde{E}[\phi](\tau_1) + G[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2} + D^\alpha[F]_{\tau_1}^{\tau_2}. \quad (21)$$

Then the above integrated local energy inequality is improved to

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} d\text{vol} \lesssim \tilde{E}[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + G[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2}. \quad (22)$$

We have shown (21), (22) for all  $\tau_2 \geq \tau_1 \geq t_0$ . We claim that these two inequalities hold for all  $\tau_2 \geq \tau_1 \geq 0$ . In fact, when  $\tau_1 \leq \tau_2 \leq t_0$ , the finite speed of propagation for wave equation [21] shows that  $\phi$  vanishes when  $r \geq R = t_0 + R_0$ . Hence we can show

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (|N| + |L|) |\bar{\partial}\phi| dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{r \leq R} |\bar{\partial}\phi|^2 dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau,$$

When considering the energy inequality (9),  $\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau$  can be absorbed by using Gronwall's inequality since  $\tau_2 \leq t_0$ . Hence we can conclude (21), (22) for all  $0 \leq \tau_1 \leq \tau_2 \leq t_0$ . For the case  $\tau_1 \leq t_0 \leq \tau_2$ , split the interval  $[\tau_1, \tau_2]$  into  $[\tau_1, t_0]$  and  $[t_0, \tau_2]$ , on which we have two separate inequalities. Combining them together, we get (21), (22). Therefore (21), (22) hold for all  $0 \leq \tau_1 \leq \tau_2$ .

We end this section by making a remark. We have used the modified energy flux  $\tilde{E}[\phi](\tau)$  instead of  $E[\phi](\tau)$  to make the above argument rigorous. We claim that the inequalities (21), (22) hold if we replace  $\tilde{E}[\phi](\tau)$  with  $E[\phi](\tau)$ . In fact, it is sufficient to consider the case when

$$\tilde{E}[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + G[2+2\alpha, 1+2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1+2\alpha, \alpha]_{\tau_1}^{\tau_2}$$

is finite. By (21), this shows that  $\tilde{E}[\phi](\tau)$  is finite for all  $\tau \in [\tau_1, \tau_2]$ . Thus Remark 1 shows that all the above statements hold if we replace  $\tilde{E}[\phi](\tau)$  with  $E[\phi](\tau)$  for  $\tau \in [\tau_1, \tau_2]$ . In the sequel, we no longer use the modified energy flux  $\tilde{E}[\phi](\tau)$  for the reason argued here.

### 3.2 p-weighted Energy inequality

We revisit the p-weighted energy inequalities developed by M. Dafermos and I. Rodnianski in [5]. Rewrite the equation (13) in null coordinates

$$-\partial_u \partial_v \psi + \Delta \psi = r(F - N - L), \quad \psi := r\phi, \quad (23)$$

where  $\Delta$  denotes the Laplacian on the sphere with radius  $r$ . Multiplying the equation by  $r^p \partial_v \psi$  and integration by parts in the region bounded by the two null hypersurfaces  $S_{\tau_1}, S_{\tau_2}$  and the hypersurface  $\{r = R\}$ , we obtain

$$\begin{aligned} & \int_{S_{\tau_2}} r^p (\partial_v \psi)^2 dv d\omega + 2 \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{p+1} (F - N - L) \partial_v \psi dv d\omega \\ & + \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{p-1} (p(\partial_v \psi)^2 + (2-p)|\nabla \psi|^2) dv d\tau d\omega + \int_{\mathcal{I}_{\tau_1}^{\tau_2}} r^p |\nabla \psi|^2 dud\tau d\omega \\ & = \int_{S_{\tau_1}} r^p (\partial_v \psi)^2 dv d\omega + \int_{\tau_1}^{\tau_2} r^p (|\nabla \psi|^2 - (\partial_v \psi)^2) d\omega d\tau|_{r=R}. \end{aligned} \quad (24)$$

We claim that we can estimate the boundary terms on  $r = R$  as follows

$$\left| \int_{\tau_1}^{\tau_2} r^p (|\nabla \psi|^2 - (\partial_v \psi)^2) d\omega d\tau \right|_{r=R} \lesssim E[\phi](\tau_1) + G[2 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1 + 2\alpha, \alpha]_{\tau_1}^{\tau_2} + D^\alpha[F]_{\tau_1}^{\tau_2}. \quad (25)$$

Since  $R$  is a fixed constant, it suffices to show (25) for  $p = 0$ . Thus take  $p = 0$  in the identity (24). The energy term on the null hypersurfaces  $S_{\tau_1}$ ,  $S_{\tau_2}$ ,  $\mathcal{I}_{\tau_1}^{\tau_2}$  can be bounded by  $\tilde{E}(\tau_2) + \tilde{E}(\tau_1)$ , which can be estimated by using the energy inequality (21). We use the improved integrated local energy estimates for  $\nabla \phi$  to bound the third term in (24). Recall that when  $r \geq R$ , we in fact have the improved lower bound

$$\frac{1}{r} \lesssim \frac{\beta((1+r)^\alpha - 1)}{r(1+r)^\alpha} - \frac{1}{(1+r)^{1+\alpha}} = \chi - \frac{1}{2}f'$$

instead of  $\frac{1}{(1+r)^{1+\alpha}}$  we have used in (8) to obtain (18). Thus we actually can show that

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{-1} |\nabla \psi|^2 dv d\omega d\tau = \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\nabla \phi|^2}{r} d\text{vol} \lesssim E[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |F - N - L| |\bar{\partial} \phi| d\text{vol}.$$

For the inhomogeneous term, notice that

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} |r(F - N - L) \partial_v \psi| dv d\omega d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{S_\tau} |F - N - L| |\bar{\partial} \phi| d\text{vol}.$$

We have already shown that this term can be bounded by the right hand side of (25) in the previous section. Thus the inequality (25) follows.

Now, to make use of the identity (24), we need to control the inhomogeneous term  $r^{p+1}(F - N - L) \partial_v \psi$  as all the other terms have a positive sign or are bounded. Due to the different structures of  $F$ ,  $N$ ,  $L$ , we discuss them separately. The most difficult term is the linear term  $r^{p+1}N(\Phi, \phi) \partial_v \psi$  satisfying the null condition. For this term, by Lemma 4, it suffices to estimate the following three terms

$$r^p |\partial \Phi| |\bar{\partial}_v \psi| |\partial_v \psi|, \quad r^p |\partial \Phi| |\phi| |\partial_v \psi|, \quad r^p |\partial_v \Phi| |\partial \psi| |\partial_v \psi|.$$

In application,  $p \in (0, 2)$ . In particular, the coefficients  $p$ ,  $2 - p$  in (24) are positive. From the decay assumptions on  $\Phi$  (see Definition 1), we estimate the first term as follows

$$\begin{aligned} 2r^p |\partial \Phi| |\bar{\partial}_v \psi| |\partial_v \psi| &\leq 2r^{p-\frac{1}{2}} (1 + (t - |x|)_+)^{-\frac{1}{2}-4\alpha} |\bar{\partial}_v \psi| |\partial_v \psi| \\ &\leq \epsilon_1 r^{p-1} |\bar{\partial}_v \psi|^2 + \frac{C}{\epsilon_1} r^p (1 + \tau)^{-1-8\alpha} |\partial_v \psi|^2, \quad \forall \epsilon_1 > 0. \end{aligned} \quad (26)$$

The first term will be absorbed if  $p > \epsilon_1$ ,  $2 - p > \epsilon_1$ , while the second term will be controlled by using Gronwall's inequality. Similarly for the second term  $r^p |\partial \Phi| |\phi| |\partial_v \psi|$ , we can show

$$2r^p |\partial \Phi| |\phi| |\partial_v \psi| \lesssim r^{p-1+3\alpha} (1 + \tau)^{-6\alpha} |\partial_v \psi|^2 + r^{p-3\alpha} (1 + \tau)^{-1-2\alpha} |\phi|^2.$$

We use interpolation to further bound the first term on the right hand side of the above inequality. Notice that

$$p \cdot \frac{5\alpha}{1 + 2\alpha} \geq p - 1 + 3\alpha, \quad p \leq 1 + 2\alpha.$$

Using Hölder's inequality and Jensen's inequality, we have

$$r^{p-1+3\alpha} (1 + \tau)^{-6\alpha} \leq (r^p (1 + \tau)^{-1-\alpha})^{\frac{5\alpha}{1+2\alpha}} \cdot ((1 + \tau)^{-\alpha})^{1-\frac{5\alpha}{1+2\alpha}} \leq \tau^{-\alpha} + r^p (1 + \tau)^{-1-\alpha}.$$

We use estimate (20) to bound  $r^{p-3\alpha}\phi^2$ . Summarizing, we can show that for  $p \leq 1 + 2\alpha$

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^p |\partial \Phi| |\phi| |\partial_v \psi| dv d\omega d\tau \\ & \lesssim \tau_1^{-\alpha} G[0, 0]_{\tau_1}^{\tau_2} + G[1 + \alpha, p]_{\tau_1}^{\tau_2} + G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \frac{\tilde{E}[\phi](\tau)}{(1 + \tau)^{1+2\alpha}} d\tau. \end{aligned} \quad (27)$$

It remains to handle the third term  $r^p |\partial_v \Phi| |\partial \psi| |\partial_v \psi|$ . We estimate this term and the linear term  $r^{p+1} L(\partial \phi) \partial_v \psi$  together due to the similar assumptions on  $\partial_v \Phi$ ,  $L^\mu(t, x)$ . The difficulty for estimating these two terms is that we are not allowed to use Cauchy-Schwartz's inequality as we did previously. However notice that the integrated local energy is expected to decay in  $\tau$  ( $(1 + \tau)^{-1-\alpha}$ ). We can put some positive weights of  $\tau$  in the integrated local energy such that it is still bounded. To start with, observe that when  $|x| \geq R \geq 1$ , we have

$$r^p |\partial_v \Phi| |\partial \psi| + r^{p+1} |L(\partial \phi)| \lesssim r^{p-1-3\alpha} |\partial \psi| + r^{p+1-1-3\alpha} |\partial \phi| \lesssim r^{p-3\alpha} |\bar{\partial} \phi|.$$

Thus we can bound

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^p |\partial_v \Phi| |\partial \psi| |\partial_v \psi| + r^{p+1} |L(\partial \phi)| |\partial_v \psi| dv d\omega d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{p-3\alpha} |\bar{\partial} \phi| |\partial_v \psi| dv d\omega d\tau \\ & \lesssim \left( \int_{\tau_1}^{\tau_2} \tau^{1-\alpha} \int_{S_\tau} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} d\text{vol} \right)^{\frac{1}{2}} (G[1 - \alpha, 2p - 1 - 5\alpha]_{\tau_1}^{\tau_2})^{\frac{1}{2}} \\ & \lesssim \frac{1}{\epsilon_2} \int_{\tau_1}^{\tau_2} \tau^{1-\alpha} \int_{S_\tau} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} d\text{vol} + \epsilon_2 (G[0, 0]_{\tau_1}^{\tau_2})^{\frac{2\alpha}{1+\alpha}} (G[1 + \alpha, p]_{\tau_1}^{\tau_2})^{\frac{1-\alpha}{1+\alpha}} \\ & \lesssim \frac{1}{\epsilon_2} \int_{\tau_1}^{\tau_2} \tau^{1-\alpha} \int_{S_\tau} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} d\text{vol} + \epsilon_2 (G[0, 0]_{\tau_1}^{\tau_2} + G[1 + \alpha, p]_{\tau_1}^{\tau_2}) \end{aligned}$$

for all positive number  $\epsilon_2 \leq 1$ . Here we have used the fact

$$2p - 1 - 5\alpha - \frac{1 - \alpha}{1 + \alpha} p \leq 0, \quad p \leq 1 + 2\alpha.$$

We now have to show that the first term on the right hand side is bounded. We rely on the following lemma.

**Lemma 6.** *Suppose  $f(\tau)$  is smooth. Then for any  $\beta \neq 0$ , we have the identity*

$$\int_{\tau_1}^{\tau_2} s^\beta f(s) ds = \beta \int_{\tau_1}^{\tau_2} \tau^{\beta-1} \int_\tau^{\tau_2} f(s) ds d\tau + \tau_1^\beta \int_{\tau_1}^{\tau_2} f(s) ds.$$

*Proof.* Let

$$F(\tau) = \int_\tau^{\tau_2} f(s) ds.$$

Integration by parts gives the lemma.  $\square$

Apply the lemma to  $\beta = 1 - \alpha$ ,  $f(\tau) = \int_{S_\tau} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} d\sigma$ . Then the integrated local energy inequality (22) implies that

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \tau^{1-\alpha} \int_{S_\tau} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} d\text{vol} \lesssim \tau_1^{-\alpha} \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \tau_1^{1-\alpha} E[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau^{-\alpha} D^\alpha[F]_\tau^{\tau_2} d\tau \\ & \quad + G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[2\alpha, \alpha]_{\tau_1}^{\tau_2} + \tau_1^{1-\alpha} D^\alpha[F]_{\tau_1}^{\tau_2}. \end{aligned}$$

Since in application only two p-weighted energy inequalities associated to  $p = 1$  and  $p = 1 + 2\alpha$  are considered, we use interpolation to bound  $\bar{G}[2\alpha, \alpha]$

$$\bar{G}[2\alpha, \alpha]_{\tau_1}^{\tau_2} \lesssim \epsilon_2 \epsilon_3 \bar{G}[2\alpha, 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_2 \epsilon_3} \tau_1^{-2\alpha} \bar{G}[0, 0]_{\tau_1}^{\tau_2}$$

for all positive  $\epsilon_3$ , where  $\epsilon_2$  is the constant appeared before.

Our ultimate goal is to derive the decay of the energy flux  $E[\phi](\tau)$  on  $\Sigma_\tau$ . The almost energy flux  $\bar{g}(0, \tau)$  on  $S_\tau$  is related to  $E[\phi](\tau)$  by the following lemma.

**Lemma 7.**

$$E[\phi](\tau) \lesssim \bar{g}(0, \tau) + 2 \int_{r \leq R} |\partial \phi|^2 + \phi^2 dx \lesssim \tilde{E}[\phi](\tau).$$

*Proof.* In fact note that

$$\begin{aligned} \bar{g}(0, \tau) + 2 \int_{r \leq R} |\partial \phi|^2 + \phi^2 dx &= \int_{S_\tau} r^2 (\partial_v \phi)^2 + \partial_v (r \phi^2) + r^2 |\nabla \phi|^2 dv d\omega + 2 \int_{r \leq R} |\partial \phi|^2 + \phi^2 dx \\ &= E[\phi](\tau) + \int_{\omega} r \phi^2 d\omega \Big|_{v_\tau}^\infty + \int_{r \leq R} |\partial \phi|^2 + 2\phi^2 dx. \end{aligned}$$

Lemma 1 and Lemma 2 imply that

$$\bar{g}(0, \tau) + 2 \int_{r \leq R} |\partial \phi|^2 + \phi^2 dx \lesssim \tilde{E}[\phi](\tau).$$

To prove the other side of the inequality, it suffices to show that

$$\int_{\omega} r \phi^2(\tau, R, \omega) d\omega \Big|_{r=R} \leq \int_{r \leq R} |\partial_r \phi|^2 + 2\phi^2 dx.$$

Without loss of generality, assume  $R \geq 2$ . Notice that

$$R^3 \int_{\omega} \phi^2(\tau, R, \omega) d\omega = \int_0^R \int_{\omega} \partial_r (r^3 \phi^2) d\omega dr \leq 3 \int_{r \leq R} \phi^2 dx + R \int_{r \leq R} |\partial_r \phi|^2 + \phi^2 dx.$$

Hence

$$R \int_{\omega} \phi^2(\tau, R, \omega) d\omega \leq \int_{r \leq R} |\partial_r \phi|^2 + 2\phi^2 dx \leq \int_{r \leq R} |\partial \phi|^2 + 2\phi^2 dx.$$

Thus the lemma holds.  $\square$

Since  $G[0, 0]_{\tau_1}^{\tau_2} \leq \bar{G}[0, 0]_{\tau_1}^{\tau_2}$ , using Lemma 7, we can control  $\bar{G}[0, 0]$  in terms of  $E[\phi](\tau)$

$$G[0, 0]_{\tau_1}^{\tau_2} \leq \bar{G}[0, 0]_{\tau_1}^{\tau_2} \leq \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau.$$

Summarizing, we can show that

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^p |\partial_v \Phi| |\partial \psi| |\partial_v \psi| + r^{p+1} |L(\partial \phi)| |\partial_v \psi| dv d\omega d\tau \\ &\lesssim \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2 \epsilon_3} \right) \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \tau_1^{1-\alpha} E[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau^{-\alpha} D^\alpha [F]_\tau^{\tau_2} d\tau \\ &\quad + \tau_1^{1-\alpha} D^\alpha [F]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_2} G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + \epsilon_3 \bar{G}[2\alpha, 2\alpha]_{\tau_1}^{\tau_2} + G[1 + \alpha, p]_{\tau_1}^{\tau_2}. \end{aligned} \tag{28}$$

Here we used the argument in the end of previous section to replace  $\tilde{E}[\phi](\tau)$  with  $E[\phi](\tau)$ . We must remark here that the implicit constants before the other terms on the right hand side of (28) may also depend on  $\epsilon_i$ . However, since  $\epsilon_i$  will be chosen to depend only on  $R, \alpha, B^{\alpha\beta}, C_0$ , the omitted dependence will not affect the argument in the sequel.

Finally, we treat the inhomogeneous term  $r^{p+1}F \cdot \partial_v \psi$  in (24). Since  $D^{2\alpha}[F]_{\tau_1}^{\tau_2}$  is expected to decay in  $\tau$ , we put some positive weights of  $\tau$  in  $D^{2\alpha}[F]_{\tau}^{\tau_2}$  and estimate it by using Lemma 6 applied to  $\beta = p - \frac{3}{2}\alpha$ ,  $f(\tau) = \int_{S_\tau} r^{1+2\alpha}|F|^2 d\sigma$ . We can show that

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{S_\tau} 2r^{p+1}F \cdot \partial_v \psi dv d\tau d\omega \\ & \leq \epsilon_4 G[p - \frac{3}{2}\alpha, 2p - 1 - 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_4} \int_{\tau_1}^{\tau_2} (1 + \tau)^{p - \frac{3}{2}\alpha} \int_{S_\tau} |F|^2 r^{1+2\alpha} d\text{vol} \\ & \lesssim \epsilon_4 G[p - \frac{3}{2}\alpha, 2p - 1 - 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_4} \int_{\tau_1}^{\tau_2} \tau^{p-1 - \frac{3}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau + \frac{1}{\epsilon_4} \tau_1^{p - \frac{3}{2}\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \end{aligned} \quad (29)$$

for any  $0 < \epsilon_4 \leq 1$  and  $\tau_2 \geq \tau_1 \geq t_0$ .

### 3.3 Proof of Proposition 1

Having controlled  $\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{p+1}(F - N - L)\partial_v \psi dv d\omega d\tau$ , we are now able to prove Proposition 1. First let

$$\epsilon_1 = \frac{1 - 2\alpha}{2A}, \quad A = \max\{|B^{\mu\nu}|\}.$$

Hence for  $p = 1$  or  $1 + 2\alpha$ , the third term in (24) dominates the first term on the right hand side of (26). Set  $p = 1 + 2\alpha$  in (24) and  $\epsilon_4 = 1$  in (29). Combining the estimates (25), (26), (27), (28), we infer that

$$\begin{aligned} g(1 + 2\alpha, \tau_2) + \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} & \lesssim g(1 + 2\alpha, \tau_1) + \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2 \epsilon_3} \right) \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \\ & \quad + \tau_1^{1-\alpha} E[\phi](\tau_1) + \tau_1^{1+\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \epsilon_3 \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_2} G[1 + \frac{1}{2}\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2}. \end{aligned}$$

Now suppose the implicit constant before  $\epsilon_3 \bar{G}[0, 2\alpha]$  is  $C_1$ , which is independent of  $\epsilon_2, \epsilon_3$ . Take

$$\epsilon_3 = \frac{1}{2C_1}.$$

We remark here that we can choose different  $\epsilon_i$  for different values of  $p$ . In particular, we conclude that  $\epsilon_3 \bar{G}[0, 2\alpha]$  can be absorbed by the left hand side. Then apply Gronwall's inequality (Lemma 5). We can control the last term  $\frac{1}{\epsilon_2} G[1 + \frac{1}{2}\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2}$  and conclude that

$$\begin{aligned} g(1 + 2\alpha, \tau_2) + \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} & \lesssim g(1 + 2\alpha, \tau_1) + \tau_1^{1+\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \\ & \quad + \exp\left(\frac{2\tau_1^{-\frac{1}{2}\alpha}}{\alpha\epsilon_2}\right) \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2} \right) \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \tau_1^{1-\alpha} E[\phi](\tau_1). \end{aligned} \quad (30)$$

The integral of the energy on the right hand side can be estimated when we combine (30) with the  $p$ -weighted energy inequality for  $p = 1$ .

Now take  $p = 1$  in (24). First, we use interpolation to estimate the first term  $G[1 - \frac{3}{2}\alpha, 1 - 2\alpha]_{\tau_1}^{\tau_2}$  on the right hand side of (29)

$$G[1 - \frac{3}{2}\alpha, 1 - 2\alpha]_{\tau_1}^{\tau_2} \leq \left( G[1 + \frac{1}{2}\alpha, 1]_{\tau_1}^{\tau_2} \right)^{1-2\alpha} (G[0, 0]_{\tau_1}^{\tau_2})^{2\alpha} \leq G[1 + \frac{1}{2}\alpha, 1]_{\tau_1}^{\tau_2} + G[0, 0]_{\tau_1}^{\tau_2}.$$

To retrieve the full energy  $E[\phi](\tau)$  from  $\bar{g}(0, \tau)$ , by Lemma 7, add

$$2 \int_{\tau_1}^{\tau_2} \int_{r \leq R} |\partial \phi|^2 + \phi^2 dx d\tau$$



to both sides of (24). Then the integrated local energy estimate (22) restricted to the region  $r \leq R$  and Gronwall's inequality imply that

$$g(1, \tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau \lesssim g(1, \tau_1) + \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \epsilon_4 + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2} \right) \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \int_{\tau_1}^{\tau_2} \tau^{-\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \\ + \tau_1^{1-\alpha} E[\phi](\tau_1) + \tau_1^{1-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \tau_1^{-2\alpha} \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_2} G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2},$$

where we choose  $\epsilon_3 = 1$ . Assume the implicit constant before  $\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau$  in the above inequality is  $C_3$ , which is independent of  $\epsilon_2$  and  $\epsilon_4$ . Then take

$$\epsilon_4 = \frac{1}{2C_3}.$$

We get

$$g(1, \tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau \lesssim g(1, \tau_1) + \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2} \right) \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + \int_{\tau_1}^{\tau_2} \tau^{-\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \\ + \tau_1^{1-\alpha} E[\phi](\tau_1) + \tau_1^{1-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \tau_1^{-2\alpha} \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} + \frac{1}{\epsilon_2} G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2}. \quad (31)$$

Now let  $C_4$  be the implicit constant before  $\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau$  in both (30) and (31), which is independent of  $\epsilon_2$ . Then let

$$\epsilon_2 = \frac{1}{4C_4}$$

and choose a constant  $T_0 \geq t_0$  such that

$$T_0^{-\frac{1}{2}\alpha} \leq \frac{\alpha}{2} \epsilon_2.$$

In particular, for  $\tau_1 \geq T_0$ , we have

$$C_4 \left( \epsilon_2 + \frac{\tau_1^{-\alpha}}{\epsilon_2} + \frac{\tau_1^{-2\alpha}}{\epsilon_2^2} \right) \leq C_4 \left( \epsilon_2 + \frac{T_0^{-\alpha}}{\epsilon_2} + \frac{T_0^{-2\alpha}}{\epsilon_2^2} \right) \leq \frac{1}{2}.$$

We combine (30) and (31) together to control  $\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau$ . For  $\tau_2 \geq \tau_1 \geq T_0$ , we first estimate  $\bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2}$ ,  $G[1 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2}$  in (31) by using (30). Then combining all them together, we can show that the coefficient of  $\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau$  on the right hand side can be bounded by

$$\frac{1}{2} + C_4 T_0^{-2\alpha} \frac{e}{2} + \frac{C_4 T_0^{-2\alpha}}{\epsilon_2} \frac{e}{2\alpha} \frac{e}{2} < \frac{3}{4}.$$

Thus  $\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau$  can be absorbed and we can conclude that

$$g(1, \tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau \lesssim g(1, \tau_1) + \tau_1^{1-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \tau_1^{1-\alpha} E[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau^{-\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \\ + \tau_1^{-2\alpha} \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau + \tau_1^{-2\alpha} g(1 + 2\alpha, \tau_1),$$

which, in turn, improves (30) to

$$g(1 + 2\alpha, \tau_2) + \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} \lesssim g(1 + 2\alpha, \tau_1) + \tau_1^{1-\alpha} E[\phi](\tau_1) + \tau_1^{1+\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau.$$

This proves (16) and (17) for all  $\tau_2 \geq \tau_1 \geq T_0$ .

For  $t_0 \leq \tau_1 \leq \tau_2 \leq T_0$ , we make use of the boundedness of  $\tau$ . Let  $\epsilon_2 = 1$ . Inequality (30) shows that

$$g(1 + 2\alpha, \tau_2) + \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} \lesssim g(1 + 2\alpha, \tau_1) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + D^{2\alpha}[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1).$$

By Lemma 7, we have

$$\bar{G}[1 + 2\alpha, \alpha]_{\tau_1}^{\tau_2} \lesssim \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[0, 0]_{\tau_1}^{\tau_2} \lesssim g(1 + 2\alpha, \tau_1) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + D^{2\alpha}[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1).$$

Combining with the energy inequality (21), we obtain

$$E[\phi](\tau_2) \lesssim g(1 + 2\alpha, \tau_1) + \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau + D^{2\alpha}[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1).$$

Thus Gronwall's inequality indicates that

$$\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau \lesssim g(1 + 2\alpha, \tau_1) + D^{2\alpha}[F]_{\tau_1}^{\tau_2} + E[\phi](\tau_1)$$

as  $\tau_1 \leq \tau_2 \leq T_0$ . Hence (16) and (17) follow from (30), (31).

For  $\tau_1 \leq \tau_2 \leq t_0$ , the finite speed of propagation for wave equation [21] shows that  $g(p, \tau)$  vanishes. Thus (16), (17) hold. For general  $\tau_2 \geq \tau_1 \geq 0$ , divide the interval  $[\tau_1, \tau_2]$  into three (possibly two) such intervals:  $[\tau_1, t_0]$ ,  $[t_0, T_0]$  and  $[T_0, \tau_2]$ . Then (16), (17) follow by combining those three (or two) inequalities together. This completes the proof for (16), (17).

Having proven (16) and (17), we can improve the integrated local energy inequality (21) and the energy inequality (22) as follows: Integrate (17) from  $\tau_1$  to  $\tau_2$ . We obtain

$$\begin{aligned} G[2 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + \bar{G}[1 + 2\alpha, \alpha]_{\tau_1}^{\tau_2} &\leq G[2 + 2\alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} + (1 + \tau_1)^{-1-2\alpha} \bar{G}[0, 2\alpha]_{\tau_1}^{\tau_2} \\ &\lesssim E[\phi](\tau_1) + \tau_1^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} + (1 + \tau_1)^{-1-2\alpha} \left( g(1 + 2\alpha, \tau_2) + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \right), \end{aligned}$$

which, together with (21), (22), implies (14), (15). We thus finished proof for Proposition 1.

To show Corollary 2, take  $p = 1 + 2\alpha$  in (20). Interpolation shows that

$$\int_{S_\tau} r^{2\alpha} \phi^2 dv d\omega \lesssim E[\phi](\tau) + E[\phi](\tau)^{1-\frac{2\alpha}{1-\alpha}} g(1 + 2\alpha, \tau)^{\frac{2\alpha}{1-\alpha}}.$$

Using Jensen's inequality, we have

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{r^{2\alpha} \phi^2}{(1 + \tau)^{1+8\alpha}} dv d\omega d\tau \lesssim \int_{\tau_1}^{\tau_2} \frac{E[\phi](\tau)}{(1 + \tau)^{1+2\alpha}} d\tau + G[4 - \alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2}.$$

Therefore for  $\tau_2 \geq \tau_1 \geq t_0$ , Lemma 4 and Proposition 1 imply that

$$\begin{aligned} D^{2\alpha}[N]_{\tau_1}^{\tau_2} &= \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |B^{\alpha\beta} \partial_\alpha \Phi \cdot \partial_\beta \phi|^2 (1 + r)^{1+2\alpha} d\text{vol} \\ &\lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1 + r)^{1+\alpha}} d\text{vol} + \int_{\tau_1}^{\tau_2} \int_{S_\tau} |rN|^2 r^{1+2\alpha} dv d\omega d\tau \\ &\lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1 + r)^{1+\alpha}} d\text{vol} + \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{r^{2\alpha} |\bar{\partial}_v \psi|^2}{(1 + \tau)^{1+8\alpha}} + \frac{|\partial\psi|^2}{(1 + r)^{1+4\alpha}} + \frac{r^{2\alpha} \phi^2}{(1 + \tau)^{1+8\alpha}} dv d\omega d\tau \\ &\lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1 + r)^{1+\alpha}} d\text{vol} + \bar{G}[1 + 8\alpha, 2\alpha]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \frac{E[\phi](\tau)}{(1 + \tau)^{1+2\alpha}} d\tau + G[2 + \alpha, 1 + 2\alpha]_{\tau_1}^{\tau_2} \\ &\lesssim E[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2} + (1 + \tau_1)^{-\alpha} D^{2\alpha}[F]_{\tau_1}^{\tau_2} \\ &\quad + (1 + \tau_1)^{-1-2\alpha} \left( g(1 + 2\alpha, \tau_2) + \int_{\tau_1}^{\tau_2} \tau^{\frac{1}{2}\alpha} D^{2\alpha}[F]_{\tau}^{\tau_2} d\tau \right). \end{aligned}$$

For  $\tau_1 \leq \tau_2 \leq t_0$ , notice that

$$D^{2\alpha}[N]_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau.$$

For the linear terms  $L(\partial\phi)$ , we can show

$$\begin{aligned} D^{2\alpha}[L]_{\tau_1}^{\tau_2} &= \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |L(\partial\phi)|^2 (1+r)^{1+2\alpha} d\text{vol} \lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} (1+r)^{-2-6\alpha} |\partial\phi|^2 (1+r)^{1+2\alpha} d\text{vol} \\ &\lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} d\text{vol}. \end{aligned}$$

The corollary then follows from (14) and (15).

## 4 Decay of the Solution

Under appropriate assumptions on the inhomogeneous term  $F$ , Proposition 1 leads to the decay of the energy flux  $E[\phi](\tau)$ . After commuting the equation with the vector fields  $Z$ , we obtain the pointwise decay of the solution outside the cylinder  $\{(t, x) | |x| \leq R\}$  by using Sobolev embedding and inside the cylinder by using elliptic estimates.

**Proposition 2.** *Suppose there is a constant  $C_1$  such that*

$$D^{2\alpha}[F]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

*Then for solution  $\phi$  of the linear wave equation (13), we have energy flux decay*

$$E[\phi](\tau) \lesssim (\epsilon^2 E_0 + C_1)(1+\tau)^{-1-\alpha}.$$

*Proof.* Since the initial data are supported in the region  $\{|x| \leq R_0 \leq R\}$ , the finite speed of propagation shows that  $g(1+2\alpha, 0)$  vanishes. Take  $\tau_1 = 0$  in (17). We get

$$g(1+2\alpha, \tau) = \int_{S_\tau} r^{1+2\alpha} (\partial_v \psi)^2 dv d\omega \lesssim C_1 + \epsilon^2 E_0 \quad (32)$$

and

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha} (\partial_v \psi)^2 dv d\omega d\tau \leq \bar{G}[0, 2\alpha]_0^{\tau_2} \lesssim C_1 + \epsilon^2 E_0. \quad (33)$$

We claim that we can choose a dyadic sequence  $\{\tau_n \rightarrow \infty\}$  such that

$$\int_{S_{\tau_n}} r^{2\alpha} (\partial_v \psi)^2 dv d\omega \leq (1+\tau_n)^{-1} (C_1 + \epsilon^2 E_0), \quad (34)$$

where  $\tau_n$  satisfies the inequality  $\gamma^{-2}\tau_n \leq \tau_{n-1} \leq \gamma^2\tau_n$  for some large constant  $\gamma$ . In fact, there exists  $\tau_n \in [\gamma^n, \gamma^{n+1}]$  such that (34) holds. Otherwise

$$\int_{\gamma^n}^{\gamma^{n+1}} \int_{S_\tau} r^{2\alpha} (\partial_v \psi)^2 dv d\omega d\tau \geq \ln \gamma (\epsilon^2 E_0 + C_1),$$

which contradicts to (33) if  $\gamma$  is large enough.

Take  $\tau = \tau_n$  in (32). Interpolate with (34). We obtain

$$\int_{S_{\tau_n}} r (\partial_v \psi)^2 dv d\omega \lesssim (1+\tau_n)^{-2\alpha} (\epsilon^2 E_0 + C_1).$$

Then the inequality (16) implies that for  $\tau \geq \tau_n$

$$\int_{S_\tau} r(\partial_v \psi)^2 dv d\omega + \int_{\tau_n}^\tau E[\phi](s) ds \lesssim (1 + \tau_n)^{-2\alpha} (\epsilon^2 E_0 + C_1) + \tau_n^{1-\alpha} E[\phi](\tau_n). \quad (35)$$

On the other hand the energy inequality (15) shows that for all  $s \leq \tau$

$$E[\phi](\tau) \lesssim E[\phi](s) + (1 + s)^{-1-\alpha} (\epsilon^2 E_0 + C_1).$$

In particular

$$E[\phi](\tau_1) \lesssim E[\phi](0) + \epsilon^2 E_0 + C_1 \lesssim \epsilon^2 E_0 + C_1.$$

By (35), we have

$$(\tau - \tau_n)E[\phi](\tau) - \int_{\tau_n}^\tau (1 + s)^{-1-\alpha} (\epsilon^2 E_0 + C_1) ds \lesssim (1 + \tau_n)^{-2\alpha} (\epsilon^2 E_0 + C_1) + \tau_n^{1-\alpha} E[\phi](\tau_n). \quad (36)$$

In particular for  $n = 1$

$$E[\phi](\tau) \lesssim (1 + \tau)^{-1} (\epsilon^2 E_0 + C_1).$$

Let  $\tau = \tau_{n+1}$  in (36). We obtain

$$(\tau_{n+1} - \tau_n)E[\phi](\tau_{n+1}) \lesssim (1 + \tau_n)^{-\alpha} (\epsilon^2 E_0 + C_1).$$

Since  $\tau_n$  are dyadic, we have

$$E[\phi](\tau_n) \lesssim \tau_n^{-1-\alpha} (\epsilon^2 E_0 + C_1), \quad \forall n.$$

Finally, for  $\tau \in [\tau_n, \tau_{n+1}]$ , we can show

$$E[\phi](\tau) \lesssim E[\phi](\tau_n) + (1 + \tau_n)^{-1-\alpha} (\epsilon^2 E_0 + C_1) \lesssim (1 + \tau_n)^{-1-\alpha} (\epsilon^2 E_0 + C_1) \lesssim (1 + \tau)^{-1-\alpha} (\epsilon^2 E_0 + C_1).$$

□

With the energy flux decay, we can obtain the decay of the spherical average of the solution.

**Corollary 3.** *Assume that there is a constant  $C_1$  such that*

$$D^{2\alpha}[F]_{\tau_1}^{\tau_2} \leq C_1(1 + \tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

*Then on the hypersurface  $S_\tau$ , we have*

$$\begin{aligned} \int_{\omega} |r\phi|^2 d\omega &\lesssim \epsilon^2 E_0 + C_1, & r \geq R, \\ \int_{\omega} r|\phi|^2 d\omega &\lesssim (1 + \tau)^{-1-\alpha} (\epsilon^2 E_0 + C_1), & r \geq R. \end{aligned}$$

*Proof.* By Proposition 2, the first inequality follows from (19) and (32). The second one follows from Lemma 1. □

In order to obtain the pointwise decay of the solution which is usually a consequence of Sobolev embedding, we need energy estimates for the derivative of the solution. For this purpose, we commute the equation with the vector fields  $\Omega$  and  $T$ . Under appropriate assumptions on the inhomogeneous term  $F$ , we hope to derive the same energy decay for  $\Omega^k T^j \phi$ . Denote

$$N(\phi_1, \phi_2) = B^{\mu\nu} \partial_\mu \phi_1 \cdot \partial_\nu \phi_2, \quad \forall \phi_1, \phi_2 \in C^\infty(\mathbb{R}^{3+1}),$$

where we recall that the constants  $B^{\alpha\beta}$  satisfy the null condition.

**Lemma 8.** *Let  $Z$  be  $\Omega$  or  $T$ . Then*

$$Z^\beta N(\phi_1, \phi_2) = \sum_{\beta_1 + \beta_2 = \beta} N(Z^{\beta_1} \phi_1, Z^{\beta_2} \phi_2).$$

*Proof.* Notice that  $[\Omega, \partial_r] = [\Omega, \partial_t] = [\Omega, \nabla] = 0$ . The lemma then follows from the fact that  $B^{\alpha\beta}$  satisfy the null condition.  $\square$

Based on Corollary 2, we are able to prove the decay of the energy flux of  $Z^\beta \phi$  after commuting the linear equation (13) with  $Z^\beta$ .

**Proposition 3.** *Assume that there is a constant  $C_1$  such that the inhomogeneous term  $F$  in (13) satisfies the following condition*

$$D^{2\alpha}[Z^\beta F]_{\tau_1}^{\tau_2} \leq C_1(1 + \tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0, \quad \forall \beta \leq \beta_0$$

*for some multiple indices  $|\beta_0| \leq 4$ . Assume  $\Phi, L^\mu(t, x)$  satisfy the conditions in Theorem 2. Then we have*

$$E[Z^\beta \phi](\tau) \lesssim (C_1 + \epsilon^2 E_0)(1 + \tau_1)^{-1-\alpha}, \quad (37)$$

$$D^{2\alpha}[N(Z^{\beta_1} \Phi, Z^\beta \phi)]_{\tau_1}^{\tau_2} + D^{2\alpha}[Z^{\beta_1} L^\mu \cdot Z^\beta \partial_\mu \phi]_{\tau_1}^{\tau_2} \lesssim (C_1 + \epsilon^2 E_0)(1 + \tau_1)^{-1-\alpha} \quad (38)$$

*for  $\forall \beta \leq \beta_0, |\beta_1| \leq 4$ .*

*Proof.* We prove the proposition by induction. When  $\beta = 0$ , (37) follows from Proposition 2. Since  $Z^{\beta_1} \Phi$  is  $(\delta, \alpha, t_0, R_1, C_0)$ -weak wave,  $\forall |\beta_1| \leq 4$ , Corollary 2 and inequality (32) imply that

$$D^{2\alpha}[N(Z^{\beta_1} \Phi, \phi)]_{\tau_1}^{\tau_2} + D^{2\alpha}[Z^{\beta_1} L^\mu \cdot \partial_\mu \phi]_{\tau_1}^{\tau_2} \lesssim (C_1 + \epsilon^2 E_0)(1 + \tau_1)^{-1-\alpha}, \quad \forall |\beta_1| \leq 4.$$

Assume that (37), (38) hold for all  $\beta' < \beta$ . Commute the equation (13) with  $Z^\beta$ . Using Lemma 8, we have the equation for  $Z^\beta \phi$

$$\square(Z^\beta \phi) + N(\Phi, Z^\beta \phi) + L(Z^\beta \phi) = Z^\beta F - \sum_{\beta_1 + \beta_2 \leq \beta, \beta_2 < \beta} N(Z^{\beta_1} \Phi, Z^{\beta_2} \phi) + Z^{\beta_1} L^\mu \cdot Z^{\beta_2} \partial_\mu \phi. \quad (39)$$

Since  $\beta_2 < \beta$ , by the induction assumptions, we get

$$D^{2\alpha} \left[ Z^\beta F - \sum_{\beta_2 < \beta} N(Z^{\beta_1} \Phi, Z^{\beta_2} \phi) + Z^{\beta_1} L^\mu \cdot Z^{\beta_2} \partial_\mu \phi \right]_{\tau_1}^{\tau_2} \lesssim (C_1 + \epsilon^2 E_0)(1 + \tau_1)^{-1-\alpha}.$$

Hence for  $Z^\beta \phi$ , inequality (37) follows from Proposition 2 and inequality (38) follows from Corollary 2 and Proposition 1.  $\square$

Since the angular momentum  $\Omega$  is vanishing for  $r = 0$ , we are not able to obtain the pointwise bound of the solution in the cylinder  $\{|x| \leq R\}$  by commuting the equation with  $\Omega$ . We instead rely on elliptic estimates and the vector  $T = \partial_t$  as commutators.

**Lemma 9.** *Assume that there is a constant  $C_1$  such that*

$$D^{2\alpha}[F]_{\tau_1}^{\tau_2} + D^{2\alpha}[\partial_t F]_{\tau_1}^{\tau_2} \leq C_1(1 + \tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

*Then for solution of the linear wave equation (13), we have*

$$\int_{r \leq R} |\partial^2 \phi|^2 dx = \sum_{\mu, \nu=0}^3 \int_{r \leq R} |\partial_{\mu\nu} \phi|^2 dx \lesssim (E_0 \epsilon^2 + C_1)(1 + \tau)^{-1-\alpha}.$$

*Proof.* We first assume that  $\tau \geq R$ . Take  $\beta_0 = (0, 1)$  in Proposition 3. We have

$$E[T^j \phi](\tau_1) + \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial} T^j \phi|^2}{(1+r)^{1+\alpha}} dx d\tau + D^{2\alpha} [T^j N(\Phi, \phi) + T^j L(\partial \phi)]_{\tau_1}^{\tau_2} \lesssim (E_0 \epsilon^2 + C_1) (1 + \tau_1)^{-1-\alpha}$$

for all  $j \leq 1$ . Using elliptic estimates [6], we can show that

$$\begin{aligned} \int_{r \leq R} |\partial^2 \phi|^2 dx &= \int_{r \leq R} \sum_{i,j=1}^3 |\partial_{ij} \phi|^2 dx + 2 \sum_{\alpha=0}^3 \int_{r \leq R} |\partial_\alpha \partial_t \phi|^2 dx \\ &\lesssim E[\partial_t \phi](\tau) + \int_{r \leq 2R} |\Delta \phi|^2 + |\phi|^2 dx \\ &\lesssim E[\partial_t \phi](\tau) + \int_{r \leq 2R} |\partial_{tt} \phi + F - N(\Phi, \phi) - L(\partial \phi)|^2 + \phi^2 dx \\ &\lesssim E[\partial_t \phi](\tau) + \int_{r \leq 2R} |\partial_{tt} \phi|^2 dx + \sum_{j \leq 1} \int_{\tau}^{\tau+R} \int_{r \leq 2R} |T^j F|^2 + |T^j \phi|^2 + |T^j N|^2 + |T^j L|^2 dx dt. \end{aligned} \quad (40)$$

Consider the region bounded by  $\Sigma_{\tau-R}$  and  $t = \tau$ . Take  $X = T$  in (7). Note that the vector field  $T$  is killing, that is  $K^T[\partial_t \phi] = 0$ . We can conclude that

$$\int_{r \leq 2R} J_\mu^T[\partial_t \phi] n^\mu d\sigma = \int_{\Sigma_{\tau-R} \cap \{t \leq \tau\}} J_\mu^T[\partial_t \phi] n^\mu d\sigma + \int_{\tau-R}^{\tau} \int_{r \leq R+t-\tau} (TN + TL - \partial_t F) \partial_{tt} \phi d\text{vol}.$$

Apply Cauchy-Schwartz inequality to the last term. We obtain

$$\begin{aligned} &\int_{\tau-R}^{\tau} \int_{r \leq R+t-\tau} |\partial_t F - TN - TL| |\partial_{tt} \phi| d\text{vol} \\ &\lesssim \int_{\tau-R}^{\tau} \int_{\Sigma_t} |\partial_t F - TN - TL|^2 (1+r)^{\alpha+1} + \frac{|\partial \partial_t \phi|^2}{(1+r)^{\alpha+1}} dx dt \\ &\lesssim \int_{\tau-R}^{\tau} \int_{\Sigma_t} \frac{|\partial \partial_t \phi|^2}{(1+r)^{1+\alpha}} dx dt + D^{2\alpha} [\partial_t F]_{\tau-R}^{\tau} + D^{2\alpha} [TN]_{\tau-R}^{\tau} + D^{2\alpha} [TL]_{\tau-R}^{\tau} \\ &\lesssim (E_0 \epsilon^2 + C_1) (1 + \tau)^{-1-\alpha}. \end{aligned}$$

Hence we can estimate

$$\begin{aligned} \int_{r \leq 2R} |\partial_{tt} \phi|^2 dx &\leq 2 \int_{r \leq 2R} J_\mu^T[\partial_t \phi] n^\mu d\sigma \lesssim E[\partial_t \phi](\tau - R) + \int_{\tau-R}^{\tau} \int_{r \leq R+t-\tau} |\partial_t F - TN - TL| |\partial_{tt} \phi| d\text{vol} \\ &\lesssim (E_0 \epsilon^2 + C_1) (1 + \tau_1)^{-1-\alpha}. \end{aligned}$$

Then from (40), we get

$$\begin{aligned} \int_{r \leq R} |\partial^2 \phi|^2 dx &\lesssim (E_0 \epsilon^2 + C_1) (1 + \tau_1)^{-1-\alpha} + \sum_{j \leq 1} \int_{\tau-R}^{\tau+R} \int_{\Sigma_t} \frac{|\bar{\partial} \phi|^2}{(1+r)^{1+\alpha}} + |T^j F|^2 + |T^j N|^2 + |T^j L|^2 \\ &\lesssim (E_0 \epsilon^2 + C_1) (1 + \tau_1)^{-1-\alpha} + \sum_{j \leq 1} D^\alpha [T^j F]_{\tau-R}^{\tau+R} + D^{2\alpha} [T^j N]_{\tau-R}^{\tau+R} + D^{2\alpha} [T^j L]_{\tau-R}^{\tau+R} \\ &\lesssim (E_0 \epsilon^2 + C_1) (1 + \tau_1)^{-1-\alpha}. \end{aligned}$$

Thus we have proven the lemma for  $\tau \geq R$ . When  $\tau \leq R$ , the finite speed of propagation shows that the solution of (13) vanishes when  $|x| \geq t + R_0$ . Thus we can replace  $\tau - R$  with 0 in the above argument. And the lemma still holds.  $\square$

A corollary of the above lemma is the following pointwise decay of the solution in the cylinder  $\{r \leq R\}$ .

**Corollary 4.** Assume that there is a constant  $C_1$  such that

$$D^{2\alpha}[F]_{\tau_1}^{\tau_2} + D^{2\alpha}[\partial_t F]_{\tau_1}^{\tau_2} \leq C_1(1 + \tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

Then for solution  $\phi$  of (13), we have

$$|\phi|^2 \lesssim (C_1 + \epsilon^2 E_0)(1 + \tau)^{-1-\alpha}, \quad r \leq R.$$

*Proof.* Using Sobolev embedding and Lemma 9, when  $|x| \leq R$ , we can estimate

$$\begin{aligned} \phi^2 &\lesssim \int_{r \leq R} \sum_{i,j=1}^3 |\partial_{ij} \phi|^2 + \phi^2 dx \\ &\lesssim \int_{r \leq R} |\partial^2 \phi|^2 dx + \int_{\tau}^{\tau+R} \int_{r \leq R} |\partial_t \phi|^2 + |\phi|^2 dx dt \\ &\lesssim (C_1 + \epsilon^2 E_0)(1 + \tau)^{-1-\alpha}, \end{aligned}$$

where the last step follows from the integrated local energy inequality (14) restricted to the region  $r \leq R$ .  $\square$

## 5 Bootstrap Argument

To solve our nonlinear problem (4), we use the standard Picard iteration process. We prove, by a bootstrap argument, that the nonlinear term  $D^{2\alpha}[F]_{\tau_1}^{\tau_2}$  decays, which leads to the decay of the solution  $\phi$ . We still denote the quadratic nonlinearity  $A^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$  of  $F$  in (4) as  $N(\phi, \phi) = A^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ , in which the constants  $A^{\mu\nu}$  satisfy the null condition.

**Proposition 4.** Suppose  $Z^k\Phi$  is  $(\delta, \alpha, t_0, R_1, C_1)$ -weak wave for all  $|k| \leq 3$ . Assume

$$|\partial^2 Z^\beta \Phi| \leq C_1, \quad \forall |\beta| \leq 1.$$

Assume the functions  $L^\mu(t, x)$ ,  $h^{\mu\nu}(t, x)$  satisfy the conditions in Theorem 2. If the nonlinearity  $F$  in (4) satisfies

$$\begin{aligned} D^{2\alpha}[Z^\beta F]_{\tau_1}^{\tau_2} &\leq 2E_0\epsilon^2(1 + \tau_1)^{-1-\alpha}, \quad \forall |\beta| \leq 3, \quad \forall \tau_2 \geq \tau_1 \geq 0, \\ \int_{r \leq R} |\nabla Z^\beta F|^2 dx &\leq 2E_0\epsilon^2(1 + \tau)^{-1-\alpha}, \quad \forall |\beta| \leq 1, \quad \forall \tau \geq 0, \end{aligned}$$

then

$$D^{2\alpha}[Z^\beta F]_{\tau_1}^{\tau_2} \lesssim E_0^2\epsilon^4(1 + \tau_1)^{-1-\alpha}, \quad \forall |\beta| \leq 3, \quad \forall \tau_2 \geq \tau_1 \geq 0, \quad (41)$$

$$\int_{r \leq R} |\nabla Z^\beta F|^2 dx \lesssim E_0^2\epsilon^4(1 + \tau)^{-1-\alpha}, \quad \forall |\beta| \leq 1, \quad \forall \tau \geq 0. \quad (42)$$

**Remark 9.** If the given function  $\Phi$  is assumed as in Theorem 2, that is,  $Z^\beta\Phi$  is  $(\delta, \alpha, t_0, R_1, C_1)$ -weak wave for all  $|\beta| \leq 4$  and  $|\partial^2 Z^\beta \Phi| \leq C_1$  for all  $|\beta| \leq 2$ , then the above proposition holds if we replace  $|\beta| \leq 3$ ,  $|\beta| \leq 1$  with  $|\beta| \leq 4$ ,  $|\beta| \leq 2$  respectively. The reason that we formulate the proposition as above is that three derivatives are the minimum to close the bootstrap argument. Four derivatives is needed to obtain  $C^2$  solution of the equation (4).

The proof for Proposition 4 is quite similar to that in [22]. For completeness, we repeat it here. Since higher order nonlinearity decays much better, we only consider the quadratic nonlinearities  $N(\phi, \phi)$  and  $Q(\partial\phi) = h^{\mu\nu}(t, x)\partial_\mu\phi\partial_\nu\phi$ . First, Lemma 8 and the assumptions on  $h^{\mu\nu}(t, x)$

$$|Z^\beta h^{\mu\nu}| \lesssim (1 + r)^{-\frac{3}{2}\alpha}, \quad \forall |\beta| \leq 4$$

imply that

$$\begin{aligned}
D^{2\alpha}[Z^\beta F]_{\tau_1}^{\tau_2} &\lesssim \sum_{\beta_1+\beta_2\leq\beta} \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} |N(Z^{\beta_1}\phi, Z^{\beta_2}\phi)|^2 (1+r)^{1+2\alpha} + |\partial Z^{\beta_1}\phi|^2 |\partial Z^{\beta_2}\phi|^2 (1+r)^{1-\alpha} dx d\tau \\
&\lesssim \sum_{\beta_1+\beta_2\leq\beta} \int_{\tau_1}^{\tau_2} \int_{r\leq R} |\partial\phi_1|^2 |\partial\phi_2|^2 dx d\tau + \sum_{\beta_1+\beta_2\leq\beta} \int_{\tau_1}^{\tau_2} \int_{S_\tau} |N(\phi_1, \phi_2)|^2 r^{1+2\alpha} dx d\tau \\
&\quad + \sum_{\beta_1+\beta_2\leq\beta} \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\partial\phi_1|^2 |\partial\phi_2|^2 (1+r)^{1-\alpha} dx d\tau,
\end{aligned} \tag{43}$$

where we denote  $\phi_1 = Z^{\beta_1}\phi$ ,  $\phi_2 = Z^{\beta_2}\phi$ . We estimate the three integrals on the right hand side of (43) separately. We use elliptic estimates as well as the extra bootstrap assumption (41) to bound the first integral. Estimates of the second integral rely on the null structure of  $N(\phi_1, \phi_2)$ . The third integral follows from the integrated local energy inequality (14).

### 5.1 Proof of the (41) in the region $r \leq R$

When  $r \leq R$ , we use elliptic estimates to obtain the pointwise bound of the solution. However since one can only obtain elliptic estimates in a smaller region, we divide the region  $r \leq R$  into two parts:  $r \leq \frac{R}{2}$  and  $r \geq \frac{R}{2}$ . In the smaller region  $r \leq \frac{R}{2}$ , we use elliptic estimates while in the region  $r \geq \frac{R}{2}$ , we use Sobolev embedding.

Recall that  $\phi_1 = Z^{\beta_1}\phi$ ,  $\phi_2 = Z^{\beta_2}\phi$ ,  $|\beta_1| + |\beta_2| \leq 3$ . Without loss of generality, assume  $|\beta_1| \leq |\beta_2|$ . In particular we have  $|\beta_1| \leq 1$ . For  $r \leq \frac{R}{2}$ , we claim that

$$|\partial\phi_1|^2 \lesssim E_0 \epsilon^2 (1+\tau)^{-1-\alpha}, \quad r \leq \frac{R}{2}. \tag{44}$$

We first verify (44) for  $\partial_t\phi_1$ . By (39),  $\partial_t\phi_1$  satisfies the following equation

$$\square(\partial_t\phi_1) + N(\Phi, \partial_t\phi_1) + L(\partial_t\phi_1) = F_1.$$

Since  $|\beta_1 + (0, 2)| \leq 3$ , estimates (38) imply that

$$D^{2\alpha}[F_1]_{\tau_1}^{\tau_2} + D^{2\alpha}[\partial_t F_1]_{\tau_1}^{\tau_2} \lesssim E_0 \epsilon^2 (1+\tau_1)^{-1-\alpha}.$$

Thus by Corollary 4, we have

$$|\partial_t\phi_1|^2 \lesssim E_0 \epsilon^2 (1+\tau)^{-1-\alpha}, \quad r \leq \frac{R}{2}.$$

For  $\nabla\phi_1$ , notice that  $\phi_1 = Z^{\beta_1}\phi$ ,  $|\beta_1| \leq 1$  and  $|\partial^2 Z^{\beta_1}\Phi| \lesssim 1$ ,  $|\partial Z^{\beta_1}L^\mu| \lesssim 1$ . Using elliptic estimates and Lemma 9, we have for  $|x| \leq \frac{R}{2}$

$$\begin{aligned}
\|\nabla\phi_1\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})}^2 &\lesssim \int_{r\leq\frac{R}{2}} \sum_{i,j=1}^3 |\partial_{ij}\nabla\phi_1|^2 + |\nabla\phi_1|^2 dx \\
&\lesssim \int_{r\leq R} |\nabla\Delta\phi_1|^2 + |\nabla\phi_1|^2 dx \\
&\lesssim \int_{r\leq R} |\nabla(\partial_{tt}\phi_1 + Z^{\beta_1}F - Z^{\beta_1}N(\Phi, \phi) - Z^{\beta_1}L(\partial\phi))|^2 + |\nabla\phi_1|^2 dx \\
&\lesssim E[\partial_{tt}\phi_1](\tau) + E[\phi_1](\tau) + \int_{r\leq R} |\nabla Z^{\beta_1}F|^2 + |\nabla^2\phi_1|^2 + |\nabla^2\phi|^2 + |\nabla\phi_1|^2 + |\nabla\phi|^2 dx \\
&\lesssim E_0 \epsilon^2 (1+\tau_1)^{-1-\alpha},
\end{aligned} \tag{45}$$



where  $B_{\frac{1}{2}R}$  denotes the ball  $\{r \leq \frac{1}{2}R\}$  in  $\mathbb{R}^3$ . Hence we have proven (44), which implies that

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{r \leq \frac{R}{2}} |\partial \phi_1|^2 |\partial \phi_2|^2 dx &\lesssim \int_{\tau_1}^{\tau_2} (1+\tau)^{-1-\alpha} E_0 \epsilon^2 \int_{r \leq \frac{R}{2}} |\partial \phi_2|^2 dx d\tau \\ &\lesssim E_0 \epsilon^2 \int_{\tau_1}^{\tau_2} (1+\tau)^{-1-\alpha} E[\phi_2](\tau) d\tau \\ &\lesssim E_0^2 \epsilon^4 (1+\tau)^{-1-\alpha}. \end{aligned}$$

In the region  $\frac{R}{2} \leq r \leq R$ , we use the angular momentum  $\Omega$ . By Sobolev embedding on the unit sphere, we have

$$\int_{\omega} |\partial \phi_1|^2 \cdot |\partial \phi_2|^2 d\omega \lesssim \sum_{1'} \int_{\omega} |\partial \phi_{1'}|^2 d\omega \cdot \int_{\omega} |\partial \phi_2|^2 d\omega, \quad (46)$$

where we still denote  $\phi_{1'} = Z^{\beta_{1'}} \phi$  for  $\beta_{1'} \leq \beta_1 + (2, 0)$ . Notice that  $|\beta_{1'}| + |\beta_2| \leq 3 + 2 = 5$ . Without loss of generality, we assume  $|\beta_2| \leq 2$ . Thus by Lemma 9, we have

$$\int_{\omega} |\partial \phi_2|^2 d\omega \lesssim \int_{\frac{R}{2} \leq r \leq R} |\partial \phi_2|^2 + |\partial_r \partial \phi_2|^2 dx \lesssim E[\phi_2](\tau) + \int_{r \leq R} |\partial^2 \phi_2|^2 dx \lesssim (1+\tau)^{-1-\alpha} E_0 \epsilon^2, \quad (47)$$

where we have used (38), (39) and the assumption  $|\beta_2| \leq 2$  to verify the conditions in Lemma 9. Since  $|\beta_{1'}| \leq 3$ , we can show that

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{\frac{1}{2}R \leq r \leq R} |\partial \phi_1|^2 |\partial \phi_2|^2 dx d\tau &\lesssim \int_{\tau_1}^{\tau_2} \int_{\frac{R}{2}}^R \int_{\omega} |\partial \phi_1|^2 |\partial \phi_2|^2 d\omega \cdot r^2 dr d\tau \\ &\lesssim \sum_{1'} \int_{\tau_1}^{\tau_2} \int_{\frac{R}{2}}^R \int_{\omega} |\partial \phi_{1'}|^2 d\omega \int_{\omega} |\partial \phi_2|^2 d\omega \cdot r^2 dr d\tau \\ &\lesssim \sum_{1'} \int_{\tau_1}^{\tau_2} (1+\tau)^{-1-\alpha} E_0 \epsilon^2 \int_{r \leq R} |\partial \phi_{1'}|^2 dx d\tau \\ &\lesssim E_0^2 \epsilon^4 (1+\tau)^{-1-\alpha}. \end{aligned}$$

Summarizing, we have shown

$$\sum_{\beta_1 + \beta_2 \leq \beta} \int_{\tau_1}^{\tau_2} \int_{r \leq R} |\partial \phi_1|^2 |\partial \phi_2|^2 dx d\tau \lesssim E_0^2 \epsilon^4 (1+\tau)^{-1-\alpha}.$$

**Remark 10.** We remark here that (47) is only true when  $r$  is bigger than a constant. That is why we need to distinguish the two cases  $r \leq \frac{1}{2}R$  and  $r \geq \frac{1}{2}R$ .

## 5.2 Proof of (42)

Note that when  $|\beta| \leq 1$  and  $\beta_1 + \beta_2 = \beta$ , we have  $\beta_1 = 0$  or  $\beta_2 = 0$ . By (46), we have

$$\begin{aligned} \int_{\omega} |\nabla Z^{\beta} F|^2 d\omega &\lesssim \int_{\omega} |\partial Z^{\beta} \phi|^2 \cdot |\partial^2 \phi|^2 + |\partial \phi|^2 |\partial^2 Z^{\beta} \phi|^2 d\omega \\ &\lesssim \sum_{|\beta'| \leq 2} \int_{\omega} |\partial Z^{\beta'} \phi|^2 d\omega \cdot \int_{\omega} |\partial^2 Z^{\beta'} \phi|^2 d\omega + \int_{\omega} |\partial Z^{\beta'} \phi|^2 d\omega \cdot \int_{\omega} |\partial^2 Z^{\beta} \phi|^2 d\omega, \end{aligned}$$

where as pointed out previously, we only have to consider the quadratic nonlinearities  $N(\phi, \phi)$ ,  $Q(\partial \phi)$ . For  $r \leq \frac{1}{2}R$ , the inequality (44) shows that

$$|\partial Z^{\beta} \phi| \lesssim \epsilon^2 E_0 (1+\tau)^{-1-\alpha}, \quad \forall |\beta| \leq 1.$$

For  $\frac{1}{2}R \leq r \leq R$ , the inequality (47) implies that

$$\int_{\omega} |\partial Z^{\beta'} \phi|^2 d\omega \lesssim \epsilon^2 E_0 (1 + \tau)^{-1-\alpha}, \quad \forall |\beta'| \leq 2.$$

On the other hand, using Lemma 9, we obtain

$$\int_{r \leq R} |\partial^2 Z^{\beta'} \phi|^2 dx \lesssim \epsilon^2 E_0 (1 + \tau)^{-1-\alpha}, \quad \forall |\beta'| \leq 2.$$

Therefore, for all  $|\beta| \leq 1$ , we can estimate

$$\int_{r \leq R} |\nabla Z^{\beta} F|^2 dx \lesssim \epsilon^2 E_0 (1 + \tau)^{-1-\alpha} \sum_{|\beta'| \leq 2} \int_{r \leq R} |\partial^2 Z^{\beta'} \phi|^2 dx \lesssim E_0^2 \epsilon^4 (1 + \tau)^{-1-\alpha}.$$

Hence we have proven (42).

### 5.3 Proof of (41) in the region $r \geq R$

We first consider the quadratic term  $N(\phi_1, \phi_2)$ . The p-weighted energy inequality is about  $\psi = r\phi$  instead of  $\phi$ . For this reason, we expand  $N(\phi_1, \phi_2)$  in terms of  $\psi$ .

**Lemma 10.** *Suppose  $N(\phi_1, \phi_2) = A^{\alpha\beta} \partial_{\alpha} \phi_1 \partial_{\beta} \phi_2$  with constants  $A^{\alpha\beta}$  satisfying the null condition. Then*

$$r^4 |N(\phi_1, \phi_2)|^2 \lesssim \phi_1^2 \phi_2^2 + \phi_1^2 \cdot r^2 \partial_r^2 \phi_2 + |\nabla \psi_1|^2 |\nabla \psi_2|^2 + |\partial_v \psi_1|^2 |\partial_u \psi_2|^2 + |\partial_u \psi_1|^2 |\partial_v \psi_2|^2, \quad (48)$$

where  $v = \frac{t+r}{2}$ ,  $u = \frac{t-r}{2}$ .

*Proof.* In fact, notice that

$$r^2 N(\phi_1, \phi_2) = \phi_1 \phi_2 + r(\phi_1 \phi_2)_r + N(\psi_1, \psi_2)$$

and

$$|N(\psi_1, \psi_2)| \lesssim |\partial_v \psi_1| \cdot |\partial_u \psi_2| + |\partial_u \psi_1| \cdot |\partial_v \psi_2| + |\nabla \psi_1| \cdot |\nabla \psi_2|.$$

Hence the lemma holds.  $\square$

To estimate the second term in (43), it suffices to handle the terms on the right hand side of (48). We estimate the first three terms in a uniform way. Let  $\Phi_1$  be  $\phi_1$  or  $\nabla \psi_1$ ;  $\Phi_2$  be  $\phi_2$ ,  $r\partial_r \phi_2$  and  $\nabla \psi_2$  respectively. Recall that  $\phi_1 = Z^{\beta_1} \phi$ ,  $\phi_2 = Z^{\beta_2} \phi$ ,  $|\beta_1| + |\beta_2| \leq 3$ . Using Sobolev embedding on the unit sphere, we have

$$\int_{\omega} |\Phi_1|^2 |\Phi_2|^2 d\omega \lesssim \sum_{1', 2'} \int_{\omega} |\Phi_{1'}|^2 d\omega \cdot \int_{\omega} |\Phi_{2'}|^2 d\omega,$$

where we let

$$\begin{cases} \beta_{1'} \leq \beta_1 + (2, 0), & \beta_{2'} = \beta_2, & \text{if } |\beta_1| \leq 1, \\ \beta_{1'} = \beta_1, & \beta_{2'} \leq \beta_2 + (2, 0), & \text{if } |\beta_2| \leq 1. \end{cases} \quad (49)$$

In particular  $|\beta_{1'}| + |\beta_{2'}| \leq 5$ . For the third case when  $\Phi_1 = \nabla \psi_1$ ,  $\Phi_2 = \nabla \psi_2$ , without loss of generality, we assume  $|\beta_{1'}| \leq 2$ . Since  $\nabla \psi_1 = \Omega \phi_1$ ,  $\Phi_{1'}$  can always be written as  $Z^{\beta} \phi$  for some  $|\beta| \leq 3$ . Thus by Corollary 3, we have

$$r^2 \int_{\omega} |\Phi_{1'}|^2 d\omega \lesssim \epsilon^2 E_0, \quad r \geq R.$$

Recall that  $\Phi_{2'} = \phi_{2'}$ ,  $r\partial_r \phi_{2'}$  or  $\nabla \psi_{2'}$ . We always have

$$\frac{|\Phi_{2'}|^2}{(1+r)^{3+\alpha}} \lesssim \frac{|\bar{\partial} \phi_{2'}|^2}{(1+r)^{1+\alpha}}.$$

Then the integrated energy inequality (14) implies that

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha-3} \Phi_1^2 \Phi_2^2 d\text{vol} &= \int_{\tau_1}^{\tau_2} \int_{v_\tau}^\infty \int_\omega r^{2\alpha-1} \Phi_1^2 \Phi_2^2 dv d\omega d\tau \\
&\lesssim \sum_{1',2'} \int_{\tau_1}^{\tau_2} \int_{v_\tau}^\infty r^{2\alpha-3} r^2 \int_\omega |\Phi_{1'}|^2 d\omega \int_\omega |\Phi_{2'}|^2 d\omega dv d\tau \\
&\lesssim \epsilon^2 E_0 \sum_{1',2'} \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\Phi_{2'}|^2}{r^{3-2\alpha}} dv d\omega d\tau \\
&\lesssim \epsilon^2 E_0 \sum_{1',2'} \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\Phi_{2'}|^2}{(1+r)^{3+\alpha}} d\text{vol} \\
&\lesssim \epsilon^4 E_0^2 (1+\tau_1)^{-1-\alpha},
\end{aligned}$$

where we recall that  $\alpha \leq \frac{1}{4}$ . We hence have estimated the first three terms in (48).

It remains to handle the last two terms  $|\partial_v \psi_1|^2 |\partial_u \psi_2|^2$ ,  $|\partial_u \psi_1|^2 |\partial_v \psi_2|^2$ . Since they are symmetric, it suffices to consider  $|\partial_u \psi_1|^2 |\partial_u \psi_2|^2$ . Recall that  $\psi_1 = r Z^{\beta_1} \phi$ ,  $\psi_2 = r Z^{\beta_2} \phi$ ,  $|\beta_1| + |\beta_2| \leq 3$ . Define  $\beta_{1'}$ ,  $\beta_{2'}$  as in (49). In particular  $|\beta_{1'}| + |\beta_{2'}| \leq 5$ . We have two cases according to  $\beta_{i'}$ ,  $i = 1, 2$ .

We first consider the case when  $|\beta_{1'}| \leq 2$ . The idea is that we bound  $|\partial_v \psi_1|$  uniformly and then control  $|\partial_u \psi_2|^2$  by the energy flux through the null hypersurface  $v = \text{constant}$ . The following lemma shows that the energy flux through  $v = \text{constant}$  is bounded.

**Lemma 11.** *Consider the region  $D = [u_1, u_2] \times [v_1, \infty) \subset S_\tau \times [\tau_1, \tau_2]$ . Under the conditions of proposition 4, we have the energy flux estimate through the hypersurface  $v = \text{const}$*

$$\int_{u_1}^{u_2} \int_\omega (\partial_u \psi_2)^2 du d\omega \lesssim \epsilon^2 E_0 (1+\tau_1)^{-1-\alpha},$$

where  $\psi_2 = r \phi_2 = r Z^{\beta_2} \phi$ .

*Proof.* Back to the energy equation (7), take  $X = T$  on the region  $D$ . We have

$$\begin{aligned}
\int_{u_1}^{u_2} J_\mu^T[\phi_2] n^\mu d\sigma + \iint_{v \geq v_1, u = u_1} J_\mu^T[\phi_2] n^\mu d\sigma &= \iint_{v \geq v_1, u = u_2} J_\mu^T[\phi_2] n^\mu d\sigma + \int_{I_{\tau_1}^{\tau_2}} J_\mu^T[\phi_2] n^\mu d\sigma \\
&\quad + \int_D \square \phi_2 \cdot \partial_t \phi_2 d\text{vol}.
\end{aligned}$$

Using the estimates (38), we conclude that

$$D^{2\alpha}[\square \phi_2]_{\tau_1}^{\tau_2} \lesssim E_0 \epsilon^2 (1+r)^{-1-\alpha}.$$

Then by the integrated local energy inequality (14) and the energy inequality (15), we can show that

$$\int_{u_1}^{u_2} \int_\omega r^2 (\partial_u \phi_2)^2 d\omega du \leq 2 \int_{u_1}^{u_2} J_\mu^T[\phi_2] n^\mu d\sigma \lesssim E_0 \epsilon^2 (1+\tau_1)^{-1-\alpha},$$

where notice that  $D \subset S_\tau \times [\tau_1, \tau_2]$ . Thus by Corollary 3, we get

$$\int_{u_1}^{u_2} \int_\omega (\partial_u \psi_2)^2 d\omega du = \int_{u_1}^{u_2} \int_\omega r^2 (\partial_u \phi_2)^2 du d\omega + \int_\omega r \phi_2^2 d\omega \Big|_{u_1}^{u_2} \lesssim E_0 \epsilon^2 (1+\tau_1)^{-1-\alpha}.$$

□

We continue our proof of (41) when  $|\beta_{1'}| \leq 2$ . Lemma 11 and Sobolev embedding on the unit sphere show that

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha-3} |\partial_v \psi_1|^2 |\partial_u \psi_2|^2 d\text{vol} \\
&= \int_{v_{\tau_1}}^\infty \int_{u_{\tau_1}}^{u(v)} r^{2\alpha-1} \int_\omega |\partial_u \psi_2|^2 |\partial_v \psi_1|^2 d\omega du dv \\
&\leq \int_{v_{\tau_1}}^\infty \int_{u_{\tau_1}}^{u(v)} |\partial_u \psi_2|^2 d\omega du \cdot \sup_u \int_\omega r^{2\alpha-1} |\partial_v \psi_1|^2 d\omega dv \\
&\lesssim \epsilon^2 E_0 (1 + \tau_1)^{-1-\alpha} \int_{v_{\tau_1}}^\infty \sup_u \int_\omega r^{2\alpha-1} |\partial_v \psi_1|^2 d\omega dv \\
&\lesssim \epsilon^2 E_0 (1 + \tau_1)^{-1-\alpha} \int_{v_{\tau_1}}^\infty \sup_u \int_\omega r^{2\alpha} |\partial_v \psi_1|^2 d\omega dv,
\end{aligned}$$

where  $\beta_{1'}, \beta_{2'}$  are defined as in (49). Now, for all  $u \in [u_{\tau_1}, u(v)]$ , we have

$$\begin{aligned}
r^{2\alpha} (\partial_v \psi_{1'})^2 &\lesssim r^{2\alpha} (\partial_v \psi_{1'})^2|_{u=u_1} + \int_{u_1}^{u_2} r^{2\alpha} (\partial_v \psi_{1'})^2 du \\
&\quad + \int_{u_1}^{u_2} r^{2\alpha} (\partial_u \partial_v \psi_{1'})^2 du + \int_{u_1}^{u_2} r^{2\alpha-1} (\partial_v \psi_{1'})^2 du \\
&\lesssim r^{2\alpha} (\partial_v \psi_{1'})^2|_{u=u_1} + \int_{u_1}^{u_2} r^{2\alpha} (\partial_v \psi_{1'})^2 du \\
&\quad + \int_{u_1}^{u_2} r^{2\alpha} (\Delta \psi_{1'})^2 + r^{2\alpha+2} |Z^{\beta_{1'}} (F - N - L)|^2 du,
\end{aligned}$$

where we use the wave equation (23) in the last step and  $u_1 = u_\tau, u_2 = u(v)$ . Integrate on the unit sphere. We obtain

$$\begin{aligned}
& \int_{v_{\tau_1}}^\infty \sup_u \int_\omega r^{2\alpha} |\partial_v \psi_{1'}|^2 d\omega dv \lesssim \int_{S_{\tau_1}} r^{2\alpha} (\partial_v \psi_{1'})^2 dv d\omega \\
&+ \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha} (\partial_v \psi_{1'})^2 + (\nabla \Omega \phi_{1'})^2 r^{2\alpha} + r^{2\alpha+2} |Z^{\beta_{1'}} (F - N - L)|^2 dv d\omega d\tau,
\end{aligned}$$

where note that  $\nabla = \frac{\Omega}{r}$ .

We claim that the above inequality can be bounded by  $\epsilon^2 E_0$  (up to a constant). In fact, the first term can be bounded by  $\epsilon^2 E_0$  by (32); the second term can be bounded by  $\epsilon^2 E_0$  by (33); the third term can be controlled by  $(1 + \tau_1)^{-1-\alpha} \epsilon^2 E_0$  by the integrated local energy inequality (14) (notice that  $|\beta_{1'}| + 1 \leq 3$  and  $2\alpha \leq 1 - \alpha$  as  $\alpha \leq \frac{1}{4}$ ); the last term can be estimated as

$$D^{2\alpha} [Z^{\beta_{1'}} F]_{\tau_1}^{\tau_2} + D^{2\alpha} [Z^{\beta_{1'}} N]_{\tau_1}^{\tau_2} + D^{2\alpha} [Z^{\beta_{1'}} L]_{\tau_1}^{\tau_2} \lesssim E_0 \epsilon^2 (1 + \tau_1)^{-1-\alpha},$$

where we use the inequality (38). Summarizing, we have shown

$$\int_{v_{\tau_1}}^\infty \sup_u \int_\omega r^{2\alpha} |\partial_v \psi_1|^2 d\omega dv \lesssim \epsilon^2 E_0.$$

In particular, for fixed  $r \geq R$ , we have

$$\int_{t_2}^{t_1} \int_\omega |\partial_v \psi_1|^2(t, r, \omega) d\omega dt \lesssim \epsilon^2 E_0, \quad \psi_1 = r Z^\beta \phi, \quad |\beta| \leq 2. \quad (50)$$

Therefore

$$\int_{\tau_1}^{\tau_2} r^{2\alpha-3} \int_{S_\tau} |\partial_v \psi_1|^2 |\partial_u \psi_2|^2 d\text{vol} \lesssim E_0^2 \epsilon^4 (1 + \tau_1)^{-1-\alpha}.$$

**When**  $|\beta_{2'}| \leq 2$ , we control  $|\partial_v \psi_1|^2$  by the energy and bound  $\partial_u \psi_2$  uniformly. Similarly, using (32), (33) and Sobolev embedding, we obtain

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha-1} |\partial_v \psi_1|^2 |\partial_u \psi_2|^2 dv d\omega d\tau \\
& \lesssim \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha-1} |\partial_v \psi_1|^2 \cdot \int_\omega |\partial_u \psi_2|^2 d\omega \quad dv d\omega d\tau \\
& \lesssim \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{3\alpha-1} |\partial_v \psi_1|^2 \cdot r^{-\alpha} \int_\omega |\partial_u \psi_2|^2 d\omega \quad dv d\omega d\tau \\
& \lesssim \epsilon^2 E_0 \int_{\tau_1}^{\tau_2} \sup_v r^{-\alpha} \int_\omega |\partial_u \psi_2|^2 d\omega d\tau,
\end{aligned} \tag{51}$$

where  $\beta_{1'}, \beta_{2'}$  are defined as in (49). For all  $v$ , we have

$$\begin{aligned}
r^{-\alpha} (\partial_u \psi_{2'})^2 & \lesssim r^{-\alpha} (\partial_u \psi_{2'})^2|_{v=v_{\tau_2}} + \left| \int_v^{v_{\tau_2}} r^{-1-\alpha} |\partial_u \psi_{2'}|^2 dv \right| \\
& \quad + 2 \left| \int_v^{v_{\tau_2}} r^{-\alpha} |\partial_u \psi_{2'} \cdot \partial_v \partial_u \psi_{2'}| dv \right| \\
& \lesssim r^{-\alpha} (\partial_u \psi_{2'})^2|_{v=v_{\tau_2}} + \int_{v_\tau}^\infty r^{-1-\alpha} |\partial_u \psi_{2'}|^2 dv \\
& \quad + \int_{v_\tau}^\infty r^{-1-\alpha} (\partial_u \psi_{2'})^2 dv + \int_{v_\tau}^\infty r^{1-\alpha} (\partial_v \partial_u \psi_{2'})^2 dv \\
& \lesssim r^{-\alpha} (\partial_u \psi_{2'})^2|_{v=v_{\tau_2}} + \int_{v_\tau}^\infty \frac{(\partial_u \psi_{2'})^2}{r^{1+\alpha}} dv \\
& \quad + \int_{v_\tau}^\infty r^{1-\alpha} (\Delta \psi_{2'})^2 dv + \int_{v_\tau}^\infty r^{3-\alpha} |Z^{\beta_{2'}} (F - N - L)|^2 dv,
\end{aligned}$$

where  $v_{\tau_2} = \frac{R+\tau_2}{2}$ . Integrate on the unit sphere. We get

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \sup_v r^{-\alpha} \int_\omega |\partial_u \psi_{2'}|^2 d\omega d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_\omega r^{-\alpha} (\partial_u \psi_{2'})^2|_{v=v_{\tau_2}} d\tau \\
& \quad + \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{(\partial_u \psi_{2'})^2}{r^{1+\alpha}} + r^{1-\alpha} (\nabla \Omega \phi_{2'})^2 + r^{3-\alpha} |Z^{\beta_{2'}} (F - N - L)|^2 dv d\omega d\tau.
\end{aligned}$$

We claim that it can be bounded by  $(1 + \tau_1)^{-1-\alpha} \epsilon^2 E_0$  up to a constant. In fact, the first term can be estimated as follows

$$\int_{\tau_1}^{\tau_2} \int_\omega r^{-\alpha} (\partial_u \psi_{2'})^2|_{v=v_{\tau_2}} d\tau = \int_{u_{\tau_1}}^{u_{\tau_2}} \int_\omega r^{-\alpha} |\partial_u \psi_{2'}|^2 d\omega du \leq \int_{u_{\tau_1}}^{u_{\tau_2}} \int_\omega |\partial_u \psi_{2'}|^2 d\omega du \lesssim (1 + \tau_1)^{-1-\alpha} \epsilon^2 E_0$$

by Lemma 11; the second and third term can also be controlled by  $(1 + \tau_1)^{-1-\alpha} \epsilon^2 E_0$  by the integrated local energy estimates (14) (notice that  $|\beta_{2'}| \leq 2$ ); the last term can be estimated as follows

$$D^{2\alpha} [Z^{\beta_{1'}} F]_{\tau_1}^{\tau_2} + D^{2\alpha} [Z^{\beta_{1'}} N]_{\tau_1}^{\tau_2} + D^{2\alpha} [Z^{\beta_{1'}} L]_{\tau_1}^{\tau_2} \lesssim E_0 \epsilon^2 (1 + \tau_1)^{-1-\alpha}$$

by the inequality (38). Hence

$$\int_{\tau_1}^{\tau_2} \sup_v r^{-\alpha} \int_\omega |\partial_u \psi_{2'}|^2 d\omega d\tau \lesssim E_0 \epsilon^2 (1 + \tau_1)^{-1-\alpha}. \tag{52}$$

Plug this into (51). We obtain

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{2\alpha-3} |\partial_v \psi_1|^2 |\partial_u \psi_2|^2 d\text{vol} \lesssim \epsilon^4 E_0^2 (1 + \tau_1)^{-1-\alpha}.$$

Therefore using lemma 10, we have shown that

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} |N(\phi_1, \phi_2)|^2 r^{1+2\alpha} d\text{vol} \lesssim \epsilon^4 E_0^2 (1 + \tau_1)^{-1-\alpha}.$$

To show (41), it remains to estimate the quadratic term  $Z^\beta Q(\partial\phi)$ . By (43), it suffices to consider the third integral in (43). Notice that

$$|\partial\phi_1| |\partial\phi_2| \lesssim r^{-2} |\partial_v \psi_1| |\partial_u \psi_2| + \sum_{\substack{|\beta_{1'}| + |\beta_{2'}| \leq 4 \\ |\beta_{1'}, \beta_{2'}| \leq 3}} |Z^{\beta_{1'}} \phi| |\partial Z^{\beta_{2'}} \phi| + (|Z\phi| + |\partial_v \phi|) |\partial Z^3 \phi|.$$

The first term has already been estimated considering that  $r^{1-\alpha} \leq r^{1+2\alpha}$  (on  $S_\tau$ ,  $r \geq R \geq 1$ ). For the second term, if  $|\beta_{1'}| \leq 1$  or  $|\beta_{2'}| \leq 1$ , then using Sobolev inequality on the unit sphere, we have

$$\int_\omega |\phi_{1'}|^2 |\partial\phi_2|^2 d\omega \lesssim \sum_{|\beta_{1''}|, |\beta_{2''}| \leq 3} \int_\omega |\phi_{1''}|^2 d\omega \cdot \int_\omega |\partial\phi_{2''}|^2 d\omega \lesssim r^{-2} \epsilon^2 E_0 \sum_{|\beta_{2''}| \leq 3} \int_\omega |\partial\phi_{2''}|^2 d\omega$$

by (19) and (32). Then using the integrated local energy inequality (14), we obtain

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} |\partial\phi_1|^2 |\partial\phi_2|^2 (1+r)^{1-\alpha} d\text{vol} \lesssim \epsilon^2 E_0 \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\partial\phi_{2''}|^2}{(1+r)^{1+\alpha}} d\text{vol} \lesssim \epsilon^4 E_0^2 (1 + \tau_1)^{-1-\alpha}.$$

If both  $|\beta_{1'}| \geq 2$ ,  $|\beta_{2'}| \geq 2$ , recall that  $|\beta_{1'}| + |\beta_{2'}| \leq |\beta| + 1 \leq 4$ . We conclude that  $|\beta_{1'}| = |\beta_{2'}| = 2$ , for which we use the embedding

$$\int_\omega |Z^2 \phi|^2 |\partial Z^2 \phi|^2 d\omega \lesssim \|Z^2 \phi\|_{H^1(S^2)}^2 \|\partial Z^2 \phi\|_{H^1(S^2)}^2 \lesssim r^{-2} \epsilon^2 E_0 \|\partial Z^2 \phi\|_{H^1(S^2)}^2.$$

Hence

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} |\partial\phi_1|^2 |\partial\phi_2|^2 (1+r)^{1-\alpha} d\text{vol} \lesssim \sum_{|\beta| \leq 1} \epsilon^2 E_0 \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\partial Z^2 \Omega^\beta \phi|^2}{(1+r)^{1+\alpha}} d\text{vol} \lesssim \epsilon^4 E_0^2 (1 + \tau_1)^{-1-\alpha}.$$

For the third term  $(|Z\phi| + |\partial_v \phi|) |\partial Z^3 \phi|$ , using the integrated local energy inequality, it suffices to show that

$$(|Z\phi| + |\partial_v \phi|)^2 (1+r)^2 \lesssim \epsilon^2 E_0.$$

In fact, by Corollary 3 and Corollary 4, we have

$$(1+r)^2 |Z\phi|^2 \lesssim \sum_{|\beta| \leq 3} \int_\omega |Z^\beta \phi| d\omega \lesssim \epsilon^2 E_0.$$

To bound  $|\partial_v \phi|$ , notice that  $r \geq R$ . Inequality (50) implies that

$$\sum_{|\beta| \leq 2} \int_{t_2}^{t_1} \int_\omega |Z^\beta \partial_v \psi| d\omega dt \lesssim \epsilon^2 E_0, \quad \psi = r\phi.$$

Then using Sobolev embedding on  $[t_1, t_2] \times S^2$ , we obtain

$$r^2 |\partial_v \phi|^2 \leq |\partial_v \psi|^2 + |\phi|^2 \lesssim \epsilon^2 E_0.$$

Therefore

$$(1+r)^2 |\partial\phi|^2 \lesssim (1+r)^2 (|Z\phi|^2 + |\partial_v \phi|^2) \lesssim \epsilon^2 E_0.$$

In sum, (41) follows from (43). And have proven Proposition 4.

**Remark 11.** We in fact can show that

$$D^{1+\alpha} [Z^\beta F]_{\tau_1}^{\tau_2} \lesssim E_0^2 \epsilon^4 (1 + \tau_1)^{-1-\alpha}.$$

However, it is sufficient to consider  $D^{2\alpha} [Z^\beta F]_{\tau_1}^{\tau_2}$  in order to close the bootstrap argument.

## 6 Proof of the Main Theorems

We used the foliation  $\Sigma_\tau$ , part of which is null, in the previous argument. However, we do not have a local existence result with respect to the foliation  $\Sigma_\tau$ . To solve the nonlinear equation (4), we use the standard Picard iteration process. Take  $\phi_{-1}(t, x) = 0$ . We solve the following linear wave equation recursively

$$\begin{cases} \square \phi_{n+1} + N(\Phi, \phi_{n+1}) + L(\partial \phi_{n+1}) = F(\partial \phi_n), \\ \phi_{n+1}(0, x) = \epsilon \phi_0(x), \partial_t \phi_{n+1}(0, x) = \epsilon \phi_1(x). \end{cases} \quad (53)$$

Now suppose the implicit constant in Proposition 4 is  $C_1$ , which, according to our notation, depends only on  $R, \alpha, t_0, C_0, A^{\alpha\beta}, B^{\alpha\beta}$ . Set

$$\epsilon_0 = \frac{1}{\sqrt{C_1 E_0}}.$$

Then for all  $\epsilon \leq \epsilon_0$ , we have

$$C_1 \epsilon^4 E_0^2 \leq \epsilon^2 E_0.$$

Thus by the continuity of  $F(\partial \phi_n)$ , we in fact have shown that the nonlinear term  $F$  satisfies

$$D^{2\alpha}[Z^\beta F(\partial \phi_n)]_{\tau_1}^{\tau_2} \leq C_1 E_0^2 \epsilon^4 (1 + \tau_1)^{-1-\alpha} \leq E_0 \epsilon^2 (1 + \tau_1)^{-1-\alpha}, \quad \forall |\beta| \leq 4, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

Therefore, Proposition 2 implies that

$$E[Z^\beta \phi_n](\tau) \lesssim E_0 \epsilon^2 (1 + \tau)^{-1-\alpha}, \quad \forall n, \quad \forall |\beta| \leq 4.$$

After using Sobolev embedding on the unit sphere, Corollary 3 and Corollary 4 indicate that

$$\begin{aligned} |Z^\beta \phi_n| &\lesssim \sqrt{E_0} \epsilon (1 + r)^{-\frac{1}{2}} (1 + |t - r + R|)^{-\frac{1}{2} - \frac{1}{2}\alpha}, \quad \forall |\beta| \leq 2, \\ |\phi_n| &\lesssim \sqrt{E_0} \epsilon (1 + r)^{-1}. \end{aligned}$$

We also need to show that  $\phi_n$  is uniformly bounded in  $C^2$ . We first show that  $\phi_n$  is bounded in  $C^1$ . When  $r \leq \frac{1}{2}R$ , estimates (44) implies that

$$|\partial Z^k \phi_n|^2 \lesssim E_0 \epsilon^2 (1 + \tau)^{-1-\alpha}, \quad \forall k \leq 2.$$

Here we have to point out that although there  $|\beta_1| \leq 1$  (due to the assumption  $|\beta_1| + |\beta_2| \leq 3$ ), the estimate holds for  $|\beta_1| \leq 2$  if we assume  $|\beta_1| + |\beta_2| \leq 4$ , see Remark 9. When  $\frac{1}{2}R \leq r \leq R$ , using (47) and Sobolev embedding on the unit sphere, we obtain the same estimates as above. For  $r \geq R$ , the inequality (52) implies that

$$\int_{\tau_1}^{\tau_2} \sup_v r^{-\alpha} \int_\omega |\partial_u \psi_n|^2 d\omega d\tau \lesssim (1 + \tau_1)^{-1-\alpha} \epsilon^2 E_0, \quad \psi_n = r Z^\beta \phi_n, \quad |\beta| \leq 3.$$

Using Sobolev embedding on  $S^2 \times [\tau_1, \tau_2]$ , we obtain

$$|r \partial_u Z^\beta \phi_n|^2 \lesssim \phi_n^2 + r^\alpha (1 + \tau)^{-1-\alpha} \epsilon^2 E_0, \quad \forall |\beta| \leq 1.$$

Recall that  $\partial_u = \partial_t - \partial_r$  and  $|\phi_n|^2, |\partial_t \phi_n|^2 \lesssim (1 + r)^{-1} (1 + \tau)^{-1-\alpha} \epsilon^2 E_0$ . We can estimate

$$|\partial_r Z^\beta \phi_n| \lesssim (1 + r)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2} - \frac{1}{2}\alpha} \sqrt{E_0} \epsilon, \quad \forall |\beta| \leq 1.$$

Since  $\nabla = \frac{\Omega}{r}$ , we have shown that outside the cylinder  $\{r \leq R\}$

$$|\partial Z^\beta \phi_n| \lesssim (1 + r)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2} - \frac{1}{2}\alpha} \sqrt{E_0} \epsilon, \quad \forall |\beta| \leq 1.$$

It remains to control  $C^2$  estimates of  $\phi_n$ . Outside the smaller cylinder  $\{r \leq \frac{1}{4}R\}$ , we use the equation (53). In fact, we can write

$$\partial_{rr} \phi_{n+1} = F(\partial \phi_n) + \partial_{tt} \phi_{n+1} - \frac{2}{r} \partial_r \phi_{n+1} - \Delta \phi_{n+1} - N(\Phi, \phi_{n+1}) - L(\partial \phi_{n+1}).$$

Since we have already shown that

$$\begin{aligned} |\partial\phi_n|, \quad |\partial_{tt}\phi_n| &\lesssim E_0\epsilon^2(1+r)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}, \\ |\Delta\phi_n| &\leq r^{-2}|\Omega^2\phi_n| \lesssim E_0\epsilon^2(1+r)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}, \end{aligned}$$

we conclude that

$$|\partial_{rr}\phi_n| \lesssim \sqrt{E_0}\epsilon(1+r)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}.$$

Hence when  $r \leq \frac{R}{4}$ , we can show that

$$|\partial^2\phi_n| \leq |\partial_{rr}\phi_n| + |\partial Z\phi_n| \lesssim \sqrt{E_0}\epsilon(1+r)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}$$

For the case when  $r \leq \frac{R}{4}$ , we rely on elliptic theory. First we have the elliptic equation for  $\phi_{n+1}$

$$\Delta\phi_{n+1} = F(\partial\phi_n) + \partial_{tt}\phi_{n+1} - N(\Phi, \phi_{n+1}) - L(\partial\phi_{n+1}).$$

Since we have shown from (45) that

$$\|\partial Z^\beta\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})}^2 \leq \|\nabla Z^\beta\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})}^2 + \|Z^{\beta+1}\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})}^2 \lesssim E_0\epsilon^2(1+\tau)^{-1-\alpha}, \quad \forall |\beta| \leq 1,$$

we can show that the right hand side of the above elliptic equation is uniformly bounded in  $C^{\frac{1}{2}}(B_{\frac{1}{2}R})$ . Then Schauder estimates [6] imply that

$$\|\phi_n\|_{C^{2,\frac{1}{2}}(B_{\frac{1}{4}R})} \lesssim E_0\epsilon^2(1+\tau)^{-1-\alpha}.$$

In particular, we have shown that

$$\sum_{|\beta| \leq 2} |\partial^\beta\phi_n| \lesssim \sqrt{E_0}\epsilon(1+r)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}.$$

Now the classical local existence theory [8] shows that there exists a time  $t^* > 0$  and a unique smooth solution  $\phi(t, x) \in C^\infty([0, t^*) \times \mathbb{R}^3)$  of equation (4). Moreover

$$\phi_n(t, x) \rightarrow \phi(t, x)$$

in  $C^\infty([0, t^*) \times \mathbb{R}^3)$ . Therefore

$$\sum_{|\beta| \leq 2} |\partial^\beta\phi| \lesssim \sqrt{E_0}\epsilon(1+r)^{-\frac{1}{2}}(1+\tau)^{-1+\frac{1}{2}\alpha}, \quad \forall (t, x) \in [0, t^*) \times \mathbb{R}^3.$$

By a theorem of Hörmander [7] that as long as the solution is bounded up to the second order derivatives, the solution exists globally. That is there exists a unique global solution  $\phi(t, x) \in C^\infty(\mathbb{R}^{3+1})$  which solves (4). In addition since

$$\phi_n(t, x) \rightarrow \phi(t, x), \quad (t, x) \in \mathbb{R}^{3+1},$$

$\phi$  obeys all the estimates of  $\phi_n$  obtained above. We thus finished the proof of Theorem 2.

To prove Theorem 1, it suffices to check that the functions  $\Phi, \mathcal{N}^\mu(\partial\Phi), \mathcal{N}^{\mu\nu}(\partial\Phi)$  satisfy the conditions in Theorem 2 that  $\Phi, L^\mu, h^{\mu\nu}$  satisfy correspondingly. In fact, notice that

$$|\partial Z^\beta \mathcal{N}^\mu(\partial\Phi)| \leq C(\mathcal{N})|\partial^2 Z^\beta \Phi|, \quad \forall |\beta| \leq 2.$$

The boundedness of  $\partial^2 Z^\beta \Phi$ , for all  $|\beta| \leq 2$ , follows from the equation for  $Z^\beta \Phi$  (similarly, express the only unknown term  $\partial_{rr} Z^\beta \Phi$  as a combination of terms with known  $L^\infty$  norm). Hence  $|\partial Z^\beta \mathcal{N}(\partial\Phi)| + |\partial^2 Z^\beta \Phi|$



is bounded by a constant depending on  $C_0$  and the nonlinearity  $\mathcal{N}$ . For the other conditions, when  $t \leq t_0$ , we have

$$|Z^\beta \mathcal{N}^\mu(\partial\Phi)| + |Z^\beta \mathcal{N}^{\mu\nu}(\partial\Phi)| \leq C(\mathcal{N}, C_0), \quad \forall |\beta| \leq 4.$$

When  $t \geq t_0$ , we can show

$$\begin{aligned} |Z^\beta \mathcal{N}^\mu| &\leq C(\mathcal{N}, C_0)(1 + |x|)^{-1-\frac{1}{2}\alpha_0}(1 + (t - |x|)_+)^{-1}, \\ |Z^\beta \mathcal{N}^{\mu\nu}(\Phi)| &\leq C(\mathcal{N}, C_0)(1 + |x|)^{-\frac{1}{2}\alpha_0}, \quad |\beta| \leq 4. \end{aligned}$$

Replace  $\alpha$  with  $\min\{\frac{\alpha_0}{6}, \alpha\}$ . Then the functions  $\mathcal{N}^\mu(\partial\Phi)$ ,  $\mathcal{N}^{\mu\nu}(\partial\Phi)$  satisfy the conditions in Theorem 2. We thus have the stability result of Theorem 1.

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